Exercise 3.1 Since X and Y only take non-negative integers, the event $\{X+Y=k\}$ equals to the following one:

$$\{X+Y=k\} = \bigcup_{i=0}^{k} (\{X=i\} \cap \{Y=k-i\}).$$

X and Y are independent, we have $P\{\{X=i\} \cap \{Y=k-i\}\} = p_i^X p_{k-i}^Y$. It immediately follows that

$$P\{X+Y=k\} = \sum_{i=0}^{k} p_i^X p_{k-i}^Y,$$

because each $\{X=i\} \cap \{Y=k-i\}$ is exclusive for each i. Assume $X \sim B(n,p)$ and $Y \sim Be(p)$. Apparently, we have

$$\begin{aligned} P\{X=k\} &= {}_nC_k p^k (1-p)^{n-k}, \\ P\{Y=1\} &= p, \ P\{Y=0\} = 1-p. \end{aligned}$$

Then, applying the above result, we obtain

$$\begin{aligned} P\{X+Y=k\} &= p_k^X p_0^Y + p_{k-1}^X p_1^Y \\ &= {}_nC_k p^k (1-p)^{n-k} \times (1-p) + {}_nC_{k-1} p^{k-1} (1-p)^{n-k+1} \times p \\ &= ({}_nC_k + {}_nC_{k-1}) p^k (1-p)^{n-k+1}. \end{aligned}$$

Noting

$$\begin{aligned} {}_nC_k + {}_nC_{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{n!(n-k+1+k)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} \\ &= {}_{n+1}C_k, \end{aligned}$$

we have $P\{X+Y=k\} = {}_{n+1}C_k p^k (1-p)^{n+1-k}$, which implies $X+Y \sim B(n+1,p)$.

Exercise 3.2 Since, X and Y are independent, we get

$$E[\mathrm{e}^{t(X+Y)}] = E[\mathrm{e}^{tX}]E[\mathrm{e}^{tY}] = (p\mathrm{e}^t + (1-p))^n (p\mathrm{e}^t + (1-p))^m = (p\mathrm{e}^t + (1-p))^{n+m}.$$

Therefore, $X+Y \sim B(n+m,p)$.

Exercise 3.3 We have $m_X(t) = \exp(\lambda(\mathrm{e}^t - 1))$ and $m_Y(t) = \exp(\mu(\mathrm{e}^t - 1))$. Hence,

$$\begin{aligned} m_{X+Y}(t) &= E(\mathrm{e}^{(X+Y)t}) = E(\mathrm{e}^{Xt} e^{Yt}) = E(\mathrm{e}^{Xt})E(\mathrm{e}^{Yt}) \\ &= m_X(t) m_Y(t) = \exp(\lambda(\mathrm{e}^t - 1) + \mu(\mathrm{e}^t - 1)) \\ &= \exp((\lambda+\mu)(\mathrm{e}^t - 1)), \end{aligned}$$

where we use the independence of X and Y. The above equation implies $X+Y \sim Poi(\lambda+\mu)$.

Exercise 3.4 Directly calculating the MGF, we get

$$E[\mathrm{e}^{tX}] = \sum_{n=1}^{\infty} \mathrm{e}^{tn} a \frac{(1-p)^n}{n}. = a \sum_{n=1}^{\infty} \frac{1}{n} \{\mathrm{e}^t (1-p)\}^n. \tag{A.3.1}$$

We have the Taylor expansion of $\log(1-x)$ around $x=0$ as

$$\log(1-x) = \log 1 + \sum_{n=1}^{\infty} \frac{(n-1)! \cdot (-1)1^{-n}}{n!} x^n = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Hence, substituting $x = \mathrm{e}^t(1-p)$ and $a = -1/\log p$ into (A.3.1) gives us

$$E[\mathrm{e}^{tX}] = a\sum_{n=1}^{\infty}\frac{1}{n}x^n = -a\log(1-x) = -a\log(1-(1-p)\mathrm{e}^t)$$
$$= \frac{\log(1-(1-p)\mathrm{e}^t)}{\log p}.$$

The property of the MGF leads to

$$E[X] = \frac{\mathrm{d}}{\mathrm{d}t}E[\mathrm{e}^{tX}]\Big|_{t=0} = \frac{-(1-p)\mathrm{e}^0}{(\log p)\{1-(1-p)\mathrm{e}^0\}} = \frac{a(1-p)}{p}.$$

Exercise 3.5 $\{T = k\}$ represents the event that the kth trial is successful after $m-1$ successes and $k-m-2$ failures. That probability is

$$P\{T = k\} = {}_{k-1}C_{m-1}p^{m-1}(1-p)^{(k-1)-(m-1)}\cdot p$$
$$= {}_{k-1}C_{k-m}p^m(1-p)^{k-m}.$$

Exercise 3.6 Since $\{X = k\} = \{T = k+m\}$, it follows from simply substituting the result of Exercise 3.5 that

$$P\{X = k\} = {}_{k-m+m}C_{k-1+m}p^m(1-p)^{k-m+m} = {}_kC_{m+k-1}p^m(1-p)^k. \tag{A.3.2}$$

Let $T = m + k = n$. Then, (A.3.2) can be rewritten as $P\{X = k\} = {}_kC_{n-1}p^{n-k}(1-p)^k$. Substituting $p = 1 - \frac{\lambda}{n}$ into this equation, we obtain

$$P(X = k) = {}_kC_{n-1}\left(1-\frac{\lambda}{n}\right)^{n-k}\left(\frac{\lambda}{n}\right)^k = \frac{(n-1)!}{(n-1-k)!k!}\left(1-\frac{\lambda}{n}\right)^{n-k}\left(\frac{\lambda}{n}\right)^k$$
$$= \frac{\lambda^k}{k!}\left\{\frac{(n-1)!}{(n-1-k)!}\frac{\left(1-\frac{\lambda}{n}\right)^n}{(n-\lambda)^k}\right\}.$$

For large n, $\log n! \approx n\log n - n$ by Stirling's forluma. Thus, the last equation can be approximated by

$$\frac{\lambda^k}{k!}\left\{\frac{(n-1)!}{(n-1-k)!}\frac{\left(1-\frac{\lambda}{n}\right)^n}{(n-\lambda)^k}\right\} \approx \frac{\lambda^k}{k!}\left\{\frac{\mathrm{e}^{(n-1)\log(n-1)-(n-1)}}{\mathrm{e}^{(n-1-k)\log(n-k-1)-(n-k-1)}}\right\}\frac{\left(1-\frac{\lambda}{n}\right)^n}{(n-\lambda)^k}$$
$$= \frac{\lambda^k}{k!}\left\{\mathrm{e}^k\frac{(n-1)^{n-1}}{(n-k-1)^{n-k-1}}\right\}\frac{\left(\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right)^{-\lambda}}{(n-\lambda)^k}$$
$$= \frac{\lambda^k}{k!}\left\{\mathrm{e}^k\left(1-\frac{k}{n-1}\right)^{-(n-1)}\left(\frac{n-k-1}{n-\lambda}\right)^k\right\}\left(\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right)^{-\lambda}.$$

Taking $n \to \infty$, we have

$$\frac{\lambda^k}{k!}\left\{\mathrm{e}^k\left(1-\frac{k}{n-1}\right)^{-(n-1)}\left(\frac{n-k-1}{n-\lambda}\right)^k\right\}\left(\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right)^{-\lambda} \to \frac{\lambda^k}{k!}\times\mathrm{e}^{-k}\times\mathrm{e}^k\times 1\times\mathrm{e}^{-\lambda}$$
$$= \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda},$$

which is (3.8) and this completes the proof.

Exercise 3.7 Let $Y \equiv \mu + \sigma X$. Then, the MGF of Y is

$$m_Y(t) = E[\mathrm{e}^{tY}] = E[\mathrm{e}^{t(\mu+\sigma X)}] = \mathrm{e}^{t\mu}\int_{-\infty}^{\infty}\mathrm{e}^{t\sigma x}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}dx$$
$$= \mathrm{e}^{t\mu}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-\sigma t)^2}{2}}\times\mathrm{e}^{\frac{(\sigma t)^2}{2}}dx$$
$$= \mathrm{e}^{t\mu+\frac{(\sigma t)^2}{2}},$$

which is the MGF of $N(\mu, \sigma)$ itself.

Exercise 3.8

(1) Let $m_Y(t)$ be the MGF of Y. Then, we have by (3.17)

$$m_Y = \exp\{\mu t + \frac{\sigma^2 t^2}{2}\}.$$

Therefore , we obtain by Proposition 2.1 that

$$\begin{aligned} E[Y] &= m_Y^{(1)}(0) = \mu, \\ V[Y] &= E[Y^2] - E^2[Y] = m_Y^{(2)}(0) - \{m_Y^{(1)}\}^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2 \end{aligned}$$

(2) The MGF of X, $m_X(t)$, is given by $m_x(t) = \mathrm{e}^{\frac{t^2}{2}}$. So we have

$$\begin{aligned} m_X^{(3)}(t) &= (3t + t^3)\mathrm{e}^{\frac{t^2}{2}}, \\ m_X^{(4)}(t) &= (3 + 6t^2 + t^4)\mathrm{e}^{\frac{t^2}{2}}, \end{aligned}$$

which shows $E[X^3] = m_X^{(3)}(0) = 0$ and $E[X^4] = m_X^{(4)}(0) = 3$.

Exercise 3.9

(i) Let $Z \equiv X + Y$. Then, the distritution function of Z, $F_{X+Y}(z)$, is given by

$$\begin{aligned} F_{X+Y}(z) &= \int\int_{x+y\le z} f_X(x) f_Y(y)\mathrm{d}x\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f_Y(y)\mathrm{d}y \right) f_X(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} F_Y(z-x) f_X(x)\mathrm{d}x. \end{aligned}$$

Differentiating the last equation with respect to z, we have

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_Y(z-x) f_X(x)\mathrm{d}x.$$

(ii) Let $\bar{X} \equiv X - \mu_X, \bar{Y} \equiv Y - \mu_Y$. Then, each density function is given by

$$\begin{aligned} f_{\bar{X}}(x) &= \frac{1}{\sqrt{2\pi}\sigma_X} \mathrm{e}^{-\frac{x^2}{2\sigma_X^2}}, \\ f_{\bar{Y}}(y) &= \frac{1}{\sqrt{2\pi}\sigma_Y} \mathrm{e}^{-\frac{y^2}{2\sigma_Y^2}}. \end{aligned}$$

Hence, we obtain the density funciton of $\bar{X} + \bar{Y}$ as

$$\begin{aligned} f_{\bar{X}+\bar{Y}}(z) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left\{ -\frac{(z-x)^2}{2\sigma_Y^2} \right\} \left\{ -\frac{x^2}{2\sigma_X^2} \right\} \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left\{ -\frac{1}{2} Q(x,z) \right\} \mathrm{d}x, \end{aligned}$$

where

$$\begin{aligned} Q(x,z) &\equiv \frac{(z-x)^2}{\sigma_Y^2} + \frac{x^2}{\sigma_X^2} \\ &= \frac{1}{\sigma_X^2\sigma_Y^2} \left[\sigma_X^2(z^2 - 2zx + x^2) + \sigma_Y^2 x^2 \right] \\ &= \frac{(\sigma_x^2 + \sigma_Y^2)}{\sigma_X^2\sigma_Y^2} \left[x^2 - 2zx\frac{\sigma_X^2}{(\sigma_x^2 + \sigma_Y^2)} + z^2 \frac{\sigma_X^2}{(\sigma_x^2 + \sigma_Y^2)} \right] \\ &= \frac{(\sigma_x^2 + \sigma_Y^2)}{\sigma_X^2\sigma_Y^2} \left[\left(x - z\frac{\sigma_X^2}{(\sigma_x^2 + \sigma_Y^2)} \right)^2 + z^2 \frac{\sigma_Y^2\sigma_X^2}{(\sigma_x^2 + \sigma_Y^2)^2} \right]. \end{aligned}$$

Therefore, we get

$$
\begin{aligned}
f_{\bar{X}+\bar{Y}}(z) =& \frac{1}{2\pi\sigma_X^2\sigma_Y^2} \exp\left\{-\frac{1}{2}\frac{z^2}{(\sigma_X^2+\sigma_Y^2)}\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{(\sigma_X^2+\sigma_Y^2)}{\sigma_X^2\sigma_Y^2}\left(x-\frac{z\sigma_X^2}{(\sigma_X^2+\sigma_Y^2)}\right)^2\right\} \mathrm{d}x \\
=& \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma_X^2+\sigma_Y^2}} \exp\left\{-\frac{1}{2}\frac{z^2}{(\sigma_X^2+\sigma_Y^2)}\right\} \\
& \times \int_{-\infty}^{\infty} \frac{\sqrt{\sigma_X^2+\sigma_Y^2}}{\sqrt{2\pi}\sigma_X^2\sigma_Y^2} \exp\left\{-\frac{1}{2}\frac{(\sigma_X^2+\sigma_Y^2)}{\sigma_X^2\sigma_Y^2}\left(x-\frac{z\sigma_X^2}{(\sigma_X^2+\sigma_Y^2)}\right)^2\right\} \mathrm{d}x \\
=& \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma_X^2+\sigma_Y^2}} \exp\left\{-\frac{1}{2}\frac{z^2}{(\sigma_X^2+\sigma_Y^2)}\right\},
\end{aligned}
$$

which implies $\bar{X}+\bar{Y} \sim N(0, \sigma_X^2+\sigma_Y^2)$. By Exersice 3.7, we can see that $X+Y \sim N(\mu_X+\mu_y, \sigma_X^2+\sigma_Y^2)$.

Exercise 3.10 The density functions of X and Y can be expressed as follows:

$$
\begin{aligned}
f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}, \\
f_Y(y) &= \frac{1}{\sqrt{2\pi}\sigma(x)} \exp\left\{-\frac{(x-\mu(x))^2}{2\sigma^2(x)}\right\}.
\end{aligned}
$$

By the result of Exercise 3.9, we can get the density funcion of $Z = X+Y$ as

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \mathrm{d}x && \text{(A.3.3)} \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma\cdot\sigma(x)} \exp\left\{-\frac{1}{2}\left[\frac{x^2}{\sigma^2}+\frac{(z-x-\mu(x))^2}{\sigma^2(x)}\right]\right\} \mathrm{d}x. && \text{(A.3.4)}
\end{aligned}
$$

Rearranging the inside of $[\cdot]$ with tiresome algebra, we obtain

$$
\frac{x^2}{\sigma^2}+\frac{(z-x-\mu_x)^2}{\sigma_x^2} = \frac{\sigma^2+\sigma_x^2}{\sigma^2\sigma_x^2}\left[\left(x-\frac{(z-\mu_x)\sigma^2}{\sigma^2+\sigma_x^2}\right)^2+\frac{(z-\mu_x)^2\sigma^2\sigma_x^2}{(\sigma^2+\sigma_x^2)^2}\right].
$$

It follows from substituting the above into (A.3.4) that

$$
\begin{aligned}
& f_Z(z) \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma\cdot\sigma(x)} \exp\left\{-\frac{1}{2}\frac{\sigma^2+\sigma^2(x)}{\sigma^2\sigma^2(x)}\left[\left(x-\frac{(z-\mu(x))\sigma^2}{\sigma^2+\sigma^2(x)}\right)^2+\frac{(z-\mu(x))^2\sigma 2\sigma^2(x)}{(\sigma^2+\sigma^2(x))^2}\right]\right\} \mathrm{d}x.
\end{aligned}
$$

Now let us consider the condition that Z follows normal distribution. Suppose $\mu(x)$ and $\sigma(x)$ do not depend on x and denote μ_Y and σ_Y respectively. Then, by the latter result of Exercise 3.9, (**??**) becomes

$$
f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma^2+\sigma_Y^2)}} \exp\left\{-\frac{(z-\mu_Y)^2}{2(\sigma^2+\sigma_Y^2)}\right\},
$$

which is a density funciton of $N(\mu_Y, \sigma^2+\sigma_Y^2)$.

Exercise 3.11

(1)

$$
\int_1^{n+1} \log y \mathrm{d}y = \sum_{k=1}^{n} \int_k^{k+1} \log y \mathrm{d}y \geq \sum_{k=1}^{n} \log k \int_k^{k+1} \mathrm{d}y = \sum_{k=1}^{n} \log k = \log n!.
$$

(2) Since

$$\begin{aligned}
\text{(LHS)} - \text{(RHS)} &= n\log h + n\mu + n^2\sigma^2 - \int_1^{n+1} \log y \mathrm{d}y \\
&= n\log h + n\mu + n^2\sigma^2 - \left([y\log y]_1^{n+1} - \int_1^{n+1} \mathrm{d}y \right) \\
&= (\log h + \mu + 1)n + \sigma^2 n^2 - (n+1)\log(n+1) \geq 0 \quad \text{(for large } n\text{)},
\end{aligned}$$

we obtain

$$n\log h + n\mu + n^2\sigma^2 \geq \int_1^{n+1} \log y \mathrm{d}y \geq \log n!.$$

(3) Considering the following inequality:

$$\frac{\mathrm{e}^{n\log h + n\mu + n^2\sigma^2}}{n!} = \frac{\mathrm{e}^{n\log h + n\mu + n^2\sigma^2}}{\mathrm{e}^{\log n!}} \geq 1,$$

we can see the series diverge to infinity.

Exercise 3.12 Before proceeding, let us have the following auxiliary result. Let $I_k(\lambda) = \int_0^\infty x^k \mathrm{e}^{-\lambda x} \mathrm{d}x$. Then, by using the integration by parts, we obtain

$$\begin{aligned}
I_k(\lambda) &= \left[x^k \left(-\frac{1}{\lambda} e^{-\lambda x} \right) \right]_0^\infty + \frac{1}{\lambda} \int_0^\infty k x^{k-1} e^{-\lambda x} \mathrm{d}x \\
&= \frac{k}{\lambda} I_{k-1}(\lambda).
\end{aligned}$$

Apparently $I_0(\lambda) = 1/\lambda$, so we can see $I_k(\lambda) = \frac{k!}{\lambda^{k+1}}$.

(1) Let $X \sim Exp(\lambda)$. Then, from (3.26),

$$\begin{aligned}
E[X] &= \int_0^\infty x\lambda \mathrm{e}^{-\lambda x} \mathrm{d}x = \lambda I_1(\lambda) = \frac{1}{\lambda}, \\
V[X] &= E[X^2] - E[x]^2 = \int_0^\infty x^2 \lambda \mathrm{e}^{-\lambda x} \mathrm{d}x - \frac{1}{\lambda^2} \\
&= \lambda I_2(\lambda) - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda} = \frac{1}{\lambda^2}.
\end{aligned}$$

Therefore, we get $\eta = \sqrt{V[X]}/E[X] = 1$.

(2) Let $X \sim E_k(\lambda)$. Then, from (3.27),

$$\begin{aligned}
E[X] &= \int_0^\infty x \frac{\lambda^k}{(k-1)!} x^{k-1} \mathrm{e}^{-\lambda x} \mathrm{d}x = \frac{\lambda^k}{(k-1)!} I_k(\lambda) = \frac{k}{\lambda}, \\
V[X] &= E[X^2] - E[X]^2 = \int_0^\infty x^2 \frac{\lambda^k}{(k-1)!} x^{k-1} \mathrm{e}^{-\lambda x} \mathrm{d}x - \left(\frac{k}{\lambda}\right)^2 = \frac{\lambda^k}{(k-1)!} I_{k+1}(\lambda) - \left(\frac{k}{\lambda}\right)^2 \\
&= \frac{k(k+1)}{\lambda^2} - \left(\frac{k}{\lambda}\right)^2 = \frac{k}{\lambda^2}.
\end{aligned}$$

Hence, $\eta = 1/\sqrt{k} \leq 1$ since $k \geq 1$.

(3) Suppose X follows hyper-exponential distribution. Then,

$$\begin{aligned}
E[X] &= \int_0^\infty x \sum_{i=1}^k p_i \lambda_i \mathrm{e}^{-\lambda_i x} \mathrm{d}x = \sum_{i=1}^k p_i \lambda_i I_1(\lambda_i) = \sum_{i=1}^k \frac{p_i}{\lambda_i}, \\
V[X] &= E[X^2] - E[X]^2 = \int_0^\infty x^2 \sum_{i=1}^k p_i \lambda_i \mathrm{e}^{-\lambda_i x} \mathrm{d}x - \left(\sum_{i=1}^k \frac{p_i}{\lambda_i} \right)^2 \\
&= \sum_{i=1}^k p_i \lambda_i I_2(\lambda_i) - \left(\sum_{i=1}^k \frac{p_i}{\lambda_i} \right)^2 = \sum_{i=1}^k \frac{2p_i}{\lambda_i^2} - \left(\sum_{i=1}^k \frac{p_i}{\lambda_i} \right)^2.
\end{aligned}$$

To prove $\eta > 1$, it suffices to show that $\sigma^2 - \mu^2 > 0$ since both variables are positive.

$$\begin{aligned}\sigma^2 - \mu^2 &= \sum_{i=1}^{k} \frac{2p_i}{\lambda_i^2} - \left(\sum_{i=1}^{k} \frac{p_i}{\lambda_i}\right)^2 - \left(\sum_{i=1}^{k} \frac{p_i}{\lambda_i}\right)^2 \\ &= 2\left[\sum_{i=1}^{k} \frac{p_i}{\lambda_i^2} - \left(\sum_{i=1}^{k} \frac{p_i}{\lambda_i}\right)^2\right] \\ &> 0.\end{aligned}$$

where the last inequality holds by the Jensen's inequality.

Exercise 3.13

$$m(t) = E[\mathrm{e}^{tX}] = \int_0^\infty \mathrm{e}^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\lambda x} \mathrm{d}x = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \mathrm{e}^{-(\lambda-t)x} \mathrm{d}x.$$

Let us change the variable as $y = (\lambda - t)x$. Then, noting that the integrating range is the same, we have

$$m(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{y}{\lambda - t}\right)^{\alpha-1} \mathrm{e}^{-y} \frac{1}{\lambda - t} \mathrm{d}y = \frac{1}{\Gamma(\alpha)} \left(\frac{\lambda}{\lambda - t}\right)^\alpha \underbrace{\int_0^\infty y^{\alpha-1} \mathrm{e}^{-y} \mathrm{d}y}_{=\Gamma(\alpha)} = \left(\frac{\lambda}{\lambda - t}\right)^\alpha,$$

and

$$\begin{aligned}m'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\lambda - t}{\lambda}\right)^{-\alpha} = -\alpha \left(\frac{\lambda - t}{\lambda}\right)^{-\alpha-1} \frac{-1}{\lambda} = \frac{\alpha}{\lambda} \left(\frac{\lambda - t}{\lambda}\right)^{-\alpha-1}, \\ m''(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\alpha}{\lambda} \left(\frac{\lambda - t}{\lambda}\right)^{-\alpha-1}\right] = \frac{\alpha(\alpha+1)}{\lambda^2} \left(\frac{\lambda - t}{\lambda}\right)^{-\alpha-2}.\end{aligned}$$

So, we obtain

$$E[X] = m'(0) = \frac{\alpha}{\lambda}, \tag{A.3.5}$$

$$V[X] = E[X^2] - E[X]^2 = m''(0) - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha}{\lambda^2}. \tag{A.3.6}$$

Exercise 3.14 First, we consider the distribution function of Y, $F(x)$. Since $\{Y \le x\} = \{1/X \le x\} = \{X \ge 1/x\}$, we have

$$F_Y(y) = P\{Y \le x\} = \int_{\frac{1}{x}}^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} \xi^{\alpha-1} \mathrm{e}^{-\lambda\xi} \mathrm{d}\xi.$$

Thus, the density function of Y is given by

$$\begin{aligned}f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \int_{\frac{1}{x}}^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} \xi^{\alpha-1} \mathrm{e}^{-\lambda\xi} \mathrm{d}\xi \\ &= -\frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{1}{x}\right)^{\alpha-1} \mathrm{e}^{-\lambda(\frac{1}{x})} \left(-\frac{1}{x^2}\right) \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \mathrm{e}^{-\lambda/x}.\end{aligned}$$

Exercise 3.15 Since

$$\begin{aligned}\{Y_n \le x\} &= \{\max\{X_1, X_2, \cdots X_n\} - \theta \log n \le x\} \\ &= \{\max\{X_1, X_2, \cdots X_n\} \le x + \theta \log n\} \\ &= \bigcap_{i=1}^{n} \{X_i \le x + \theta \log n\},\end{aligned}$$

and X_i's are IID, we have

$$
\begin{aligned}
F_n(x) = P\{Y_n \le x\} &= \prod_{i=1}^{n} P\{X_i \le x + \theta \log n\} = \left[\int_0^{x+\theta\log n} \frac{1}{\theta}\mathrm{e}^{-\frac{y}{\theta}}\mathrm{d}y\right]^n \\
&= \left(\left[-\mathrm{e}^{-\frac{y}{\theta}}\right]_{y=0}^{x+\theta\log n}\right)^n = \left(1-\mathrm{e}^{-\frac{x+\theta\log n}{\theta}}\right)^n = \left(1-\mathrm{e}^{-\frac{x}{\theta}}\times\mathrm{e}^{-\log n}\right)^n \\
&= \left(1-\frac{\mathrm{e}^{-\frac{x}{\theta}}}{n}\right)^n.
\end{aligned}
$$

And

$$
\lim_{n\to\infty} F_n(x) = \exp\left\{-\mathrm{e}^{-\frac{x}{\theta}}\right\}
$$

is obvious by the definition of e^x.

Exercise 3.16 Denote the joint density funciton of X and Y by $\xi(x,y)$ and let $\xi_X(x) \equiv \int_{-\infty}^{\infty}\mathrm{e}^{-y}\xi(x,y)\mathrm{d}y$. Then,

$$
E[f(X)\mathrm{e}^{-Y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-y}f(x)\xi(x,y)\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty} f(x)\xi_X(x)\mathrm{d}x.
$$

Now Denote the MGF of (X,Y) by η. In other words, $\eta(s,t) = E[\mathrm{e}^{sX+tY}]$. Then, we have $\eta(s,-1) = \int_{-\infty}^{\infty}\mathrm{e}^{sx}\xi_X(x)\mathrm{d}x$ and

$$
\begin{aligned}
\eta(s,-1) &= \exp\left\{sE[X] + \frac{s^2}{2}V[X] - E[Y] + \frac{1}{2}V[Y] - sC[X,Y]\right\} \\
&= E[\mathrm{e}^{-Y}]\exp\left\{s(E[X]-C[X,Y]) + \frac{s^2}{2}V[X]\right\} \\
&= E[\mathrm{e}^{-Y}]E[\mathrm{e}^{s(X-C[X,Y])}].
\end{aligned}
$$

This gives us the following result:

$$
E[\mathrm{e}^{s(X-C[X,Y])}] = \frac{\eta(s,-1)}{E[\mathrm{e}^{-Y}]} = \int_{-\infty}^{\infty}\mathrm{e}^{sx}\frac{\xi_X(x)}{E[\mathrm{e}^{-Y}]}\mathrm{d}x.
$$

The MGF determines its distribution uniquely, so $\xi_X(x)/E[\mathrm{e}^{-Y}]$ is the density function of the random variable $X - C[X,Y]$. Hence,

$$
E[f(X)\mathrm{e}^{-Y}] = E[\mathrm{e}^{-Y}]\int_{-\infty}^{\infty} f(x)\frac{\xi_X(x)}{E[\mathrm{e}^{-Y}]}\mathrm{d}x = E[\mathrm{e}^{-Y}]E[f(X-C[X,Y])]
$$

holds and this completes the proof.

Exercise 3.17 From (3.39), the mean and the variance of the annural return are given by

$$
\begin{aligned}
E[R] &= 0.4\times 0.1 + 0.6\times 0.15 = 0.13, \\
V[R] &= (0.12)^2 + (0.18)^2 + 2\times(0.12)\times(0.18)\times(-0.6) = 0.02088,
\end{aligned}
$$

where $R = 0.4X_1 + 0.6X_2$. Since the period we consider is 5 days, we have as the mean and the variance of the 5 days return as

$$
\begin{aligned}
\mu &\equiv 0.13\times\frac{5}{360}, \\
\sigma^2 &\equiv 0.02088\times\frac{5}{360}.
\end{aligned}
$$

Noting $x_{95} = 1.645$ (see Table 3.1), we obtain as VaR with confidence level 95%

$$
\begin{aligned}
z_{95} &= 1,000,000\times\left(1.645\times\sqrt{0.02088\times\frac{5}{360}} - 0.13\times\frac{5}{360}\right) \\
&= 26207.79.
\end{aligned}
$$