

Ch. 1 Solutions

1.1. a)

$$|\psi_1\rangle = 3|+\rangle + 4|-\rangle$$

To normalize, introduce an overall complex multiplicative factor and solve for this factor by imposing the normalization condition:

$$\begin{aligned} |\psi_1\rangle &= C(3|+\rangle + 4|-\rangle) \\ 1 &= \langle\psi_1|\psi_1\rangle = \{C^*(3\langle+| + 4\langle-|)\}\{C(3|+\rangle + 4|-\rangle)\} \\ &= C^*C(9\langle+|+\rangle + 12\langle+|-\rangle + 12\langle-|+\rangle + 16\langle-|-\rangle) = C^*C(25) \\ |C|^2 &= \frac{1}{25} \end{aligned}$$

Because an overall phase is physically meaningless, we choose C to be real and positive: $C = 1/5$. Hence the normalized input state is

$$|\psi_1\rangle = \frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle.$$

Likewise:

$$\begin{aligned} |\psi_2\rangle &= C(|+\rangle + 2i|-\rangle) \\ 1 &= \{C^*(\langle+| - 2i\langle-|)\}\{C(|+\rangle + 2i|-\rangle)\} = C^*C(\langle+|+\rangle + 4\langle-|-\rangle) = |C|^2 \quad (5) \\ |\psi_2\rangle &= \frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle \end{aligned}$$

and

$$\begin{aligned} |\psi_3\rangle &= C(3|+\rangle - e^{i\pi/3}|-\rangle) \\ 1 &= \{C^*(3\langle+| - e^{-i\pi/3}\langle-|)\}\{C(3|+\rangle - e^{i\pi/3}|-\rangle)\} = C^*C(9\langle+|+\rangle + 1\langle-|-\rangle) = |C|^2 \quad (10) \\ |\psi_3\rangle &= \frac{3}{\sqrt{10}}|+\rangle - \frac{1}{\sqrt{10}}e^{i\pi/3}|-\rangle \end{aligned}$$

b) The probabilities for state 1 are

$$\begin{aligned} P_{1,+} &= |\langle+|\psi_1\rangle|^2 = |\langle+|(\frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle)|^2 = |\frac{3}{5}\langle+|+\rangle + \frac{4}{5}\langle+|-\rangle|^2 = |\frac{3}{5}|^2 = \frac{9}{25} \\ P_{1,-} &= |\langle-|\psi_1\rangle|^2 = |\langle-|(\frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle)|^2 = |\frac{3}{5}\langle-|+\rangle + \frac{4}{5}\langle-|-\rangle|^2 = |\frac{4}{5}|^2 = \frac{16}{25} \end{aligned}$$

For the other axes, we get

$$\begin{aligned} P_{1,+x} &= |\langle+_x|\psi_1\rangle|^2 = |\langle\frac{1}{\sqrt{2}}\langle+| + \frac{1}{\sqrt{2}}\langle-|)(\frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{3}{5} + \frac{1}{\sqrt{2}}\frac{4}{5}|^2 = \frac{49}{50} \\ P_{1,-x} &= |\langle-_x|\psi_1\rangle|^2 = |\langle\frac{1}{\sqrt{2}}\langle+| - \frac{1}{\sqrt{2}}\langle-|)(\frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{3}{5} - \frac{1}{\sqrt{2}}\frac{4}{5}|^2 = \frac{1}{50} \end{aligned}$$

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$$P_{1,+y} = \left| {}_y\langle + | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{3}{5} | + \rangle + \frac{4}{5} | - \rangle \right) \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{3}{5} - \frac{i}{\sqrt{2}} \frac{4}{5} \right|^2 = \frac{1}{2}$$

$$P_{1,-y} = \left| {}_y\langle - | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{3}{5} | + \rangle + \frac{4}{5} | - \rangle \right) \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{3}{5} + \frac{i}{\sqrt{2}} \frac{4}{5} \right|^2 = \frac{1}{2}$$

The probabilities for state 2 are

$$P_{2,+} = \left| \langle + | \psi_2 \rangle \right|^2 = \left| \langle + | \left(\frac{1}{\sqrt{5}} | + \rangle + \frac{2i}{\sqrt{5}} | - \rangle \right) \right|^2 = \left| \frac{1}{\sqrt{5}} \right|^2 = \frac{1}{5}$$

$$P_{2,-} = \left| \langle - | \psi_2 \rangle \right|^2 = \left| \langle - | \left(\frac{1}{\sqrt{5}} | + \rangle + \frac{2i}{\sqrt{5}} | - \rangle \right) \right|^2 = \left| \frac{2i}{\sqrt{5}} \right|^2 = \frac{4}{5}$$

$$P_{2,+x} = \left| {}_x\langle + | \psi_2 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{5}} | + \rangle + \frac{2i}{\sqrt{5}} | - \rangle \right) \right|^2 = \left| \frac{1}{\sqrt{10}} + \frac{2i}{\sqrt{10}} \right|^2 = \frac{1}{2}$$

$$P_{2,-x} = \left| {}_x\langle - | \psi_2 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{5}} | + \rangle + \frac{2i}{\sqrt{5}} | - \rangle \right) \right|^2 = \left| \frac{1}{\sqrt{10}} - \frac{2i}{\sqrt{10}} \right|^2 = \frac{1}{2}$$

$$P_{2,+y} = \left| {}_y\langle + | \psi_2 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{5}} | + \rangle + \frac{2i}{\sqrt{5}} | - \rangle \right) \right|^2 = \left| \frac{1}{\sqrt{10}} + \frac{2}{\sqrt{10}} \right|^2 = \frac{9}{10}$$

$$P_{2,-y} = \left| {}_y\langle - | \psi_2 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{5}} | + \rangle + \frac{2i}{\sqrt{5}} | - \rangle \right) \right|^2 = \left| \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} \right|^2 = \frac{1}{10}$$

The probabilities for state 3 are

$$P_{3,+} = \left| \langle + | \psi_3 \rangle \right|^2 = \left| \langle + | \left(\frac{3}{\sqrt{10}} | + \rangle - \frac{1}{\sqrt{10}} e^{i\pi/3} | - \rangle \right) \right|^2 = \left| \frac{3}{\sqrt{10}} \right|^2 = \frac{9}{10}$$

$$P_{3,-} = \left| \langle - | \psi_3 \rangle \right|^2 = \left| \langle - | \left(\frac{3}{\sqrt{10}} | + \rangle - \frac{1}{\sqrt{10}} e^{i\pi/3} | - \rangle \right) \right|^2 = \left| -\frac{1}{\sqrt{10}} e^{i\pi/3} \right|^2 = \frac{1}{10}$$

$$\begin{aligned} P_{3,+x} &= \left| {}_x\langle + | \psi_3 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{3}{\sqrt{10}} | + \rangle - \frac{1}{\sqrt{10}} e^{i\pi/3} | - \rangle \right) \right|^2 \\ &= \left| \frac{3}{\sqrt{20}} - \frac{1}{\sqrt{20}} e^{i\pi/3} \right|^2 = \left(\frac{9}{20} + \frac{1}{20} - \frac{3}{20} 2 \cos \frac{\pi}{3} \right) = \frac{7}{20} \end{aligned}$$

$$\begin{aligned} P_{3,-x} &= \left| {}_x\langle - | \psi_3 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{3}{\sqrt{10}} | + \rangle - \frac{1}{\sqrt{10}} e^{i\pi/3} | - \rangle \right) \right|^2 \\ &= \left| \frac{3}{\sqrt{20}} + \frac{1}{\sqrt{20}} e^{i\pi/3} \right|^2 = \left(\frac{9}{20} + \frac{1}{20} + \frac{3}{20} 2 \cos \frac{\pi}{3} \right) = \frac{13}{20} \end{aligned}$$

$$\begin{aligned} P_{3,+y} &= \left| {}_y\langle + | \psi_3 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{3}{\sqrt{10}} | + \rangle - \frac{1}{\sqrt{10}} e^{i\pi/3} | - \rangle \right) \right|^2 \\ &= \left| \frac{3}{\sqrt{20}} + \frac{i}{\sqrt{20}} e^{i\pi/3} \right|^2 = \left(\frac{9}{20} + \frac{1}{20} - \frac{3}{20} 2 \sin \frac{\pi}{3} \right) = \frac{1}{20} (10 - 3\sqrt{3}) \cong 0.24 \end{aligned}$$

$$\begin{aligned} P_{3,-y} &= \left| {}_y\langle - | \psi_3 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{3}{\sqrt{10}} | + \rangle - \frac{1}{\sqrt{10}} e^{i\pi/3} | - \rangle \right) \right|^2 \\ &= \left| \frac{3}{\sqrt{20}} - \frac{i}{\sqrt{20}} e^{i\pi/3} \right|^2 = \left(\frac{9}{20} + \frac{1}{20} + \frac{3}{20} 2 \sin \frac{\pi}{3} \right) = \frac{1}{20} (10 + 3\sqrt{3}) \cong 0.76 \end{aligned}$$

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c) Matrix notation:

$$|\psi_1\rangle = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix}$$

d) Probabilities in matrix notation

$$P_{1,+} = |\langle + | \psi_1 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right|^2 = \left| \frac{3}{5} \right|^2 = \frac{9}{25}$$

$$P_{1,-} = |\langle - | \psi_1 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right|^2 = \left| \frac{4}{5} \right|^2 = \frac{16}{25}$$

$$P_{1,+x} = |\langle + | \psi_1 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{3}{5} + \frac{1}{\sqrt{2}} \frac{4}{5} \right|^2 = \frac{49}{50}$$

$$P_{1,-x} = |\langle - | \psi_1 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{3}{5} - \frac{1}{\sqrt{2}} \frac{4}{5} \right|^2 = \frac{1}{50}$$

$$P_{1,+y} = |\langle + | \psi_1 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{3}{5} - \frac{i}{\sqrt{2}} \frac{4}{5} \right|^2 = \frac{1}{2}$$

$$P_{1,-y} = |\langle - | \psi_1 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{3}{5} + \frac{i}{\sqrt{2}} \frac{4}{5} \right|^2 = \frac{1}{2}$$

$$P_{2,+} = |\langle + | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{5}} \right|^2 = \frac{1}{5}$$

$$P_{2,-} = |\langle - | \psi_2 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \frac{2i}{\sqrt{5}} \right|^2 = \frac{4}{5}$$

$$P_{2,+x} = |\langle + | \psi_2 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{10}} + \frac{2i}{\sqrt{10}} \right|^2 = \frac{1}{2}$$

$$P_{2,+x} = |\langle - | \psi_2 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{10}} - \frac{2i}{\sqrt{10}} \right|^2 = \frac{1}{2}$$

$$P_{2,+y} = |\langle + | \psi_2 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{10}} + \frac{2}{\sqrt{10}} \right|^2 = \frac{9}{10}$$

$$P_{2,-y} = |\langle - | \psi_2 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} \right|^2 = \frac{1}{10}$$

$$P_{3,+} = |\langle + | \psi_3 \rangle|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \frac{3}{\sqrt{10}} \right|^2 = \frac{9}{10}$$

$$P_{3,-} = |\langle - | \psi_3 \rangle|^2 = \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \frac{-e^{i\pi/3}}{\sqrt{10}} \right|^2 = \frac{1}{10}$$

$$P_{3,+x} = |\langle + | \psi_3 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \frac{3}{\sqrt{20}} - \frac{1}{\sqrt{20}} e^{i\pi/3} \right|^2 = \frac{7}{20}$$

$$P_{3,+x} = |\langle - | \psi_3 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \frac{3}{\sqrt{20}} + \frac{1}{\sqrt{20}} e^{i\pi/3} \right|^2 = \frac{13}{20}$$

$$P_{3,+y} = |\langle + | \psi_3 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \frac{3}{\sqrt{20}} + \frac{i}{\sqrt{20}} e^{i\pi/3} \right|^2 = \frac{1}{20} (10 - 3\sqrt{3}) \cong 0.24$$

$$P_{3,-y} = |\langle - | \psi_3 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix} \right|^2 = \left| \frac{3}{\sqrt{20}} - \frac{i}{\sqrt{20}} e^{i\pi/3} \right|^2 = \frac{1}{20} (10 + 3\sqrt{3}) \cong 0.76$$

Ch. 1 Solutions

1.2 a)

State 1

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle$$

$$|\phi_1\rangle = a|+\rangle + b|-\rangle$$

$$\langle\phi_1|\psi_1\rangle = 0 \Rightarrow (a^*\langle+| + b^*\langle-|)(\frac{1}{\sqrt{3}}|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|-\rangle) = 0$$

$$a^*\frac{1}{\sqrt{3}} + ib^*\frac{\sqrt{2}}{\sqrt{3}} = 0 \Rightarrow a^* = -ib^*\sqrt{2}$$

$$|a|^2 + |b|^2 = 1 \Rightarrow |a|^2 + \frac{|a|^2}{2} = 1 \Rightarrow a = \frac{\sqrt{2}}{\sqrt{3}}$$

$$|\phi_1\rangle = \frac{\sqrt{2}}{\sqrt{3}}|+\rangle - i\frac{1}{\sqrt{3}}|-\rangle$$

State 2

$$|\psi_2\rangle = \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle$$

$$|\phi_2\rangle = a|+\rangle + b|-\rangle$$

$$\langle\phi_2|\psi_2\rangle = 0 \Rightarrow (a^*\langle+| + b^*\langle-|)(\frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle) = 0$$

$$a^*\frac{1}{\sqrt{5}} - b^*\frac{2}{\sqrt{5}} = 0 \Rightarrow a^* = b^*2$$

$$|a|^2 + |b|^2 = 1 \Rightarrow |a|^2 + \frac{|a|^2}{4} = 1 \Rightarrow a = \frac{2}{\sqrt{5}}$$

$$|\phi_2\rangle = \frac{2}{\sqrt{5}}|+\rangle + \frac{1}{\sqrt{5}}|-\rangle$$

State 3

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle + e^{i\pi/4}\frac{1}{\sqrt{2}}|-\rangle$$

$$|\phi_3\rangle = a|+\rangle + b|-\rangle$$

$$\langle\phi_3|\psi_3\rangle = 0 \Rightarrow (a^*\langle+| + b^*\langle-|)(\frac{1}{\sqrt{2}}|+\rangle + e^{i\pi/4}\frac{1}{\sqrt{2}}|-\rangle) = 0$$

$$a^*\frac{1}{\sqrt{2}} + e^{i\pi/4}b^*\frac{1}{\sqrt{2}} = 0 \Rightarrow a^* = -e^{i\pi/4}b^* \Rightarrow b = -ae^{i\pi/4}$$

$$|a|^2 + |b|^2 = 1 \Rightarrow |a|^2 + |a|^2 = 1 \Rightarrow a = \frac{1}{\sqrt{2}}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle - e^{i\pi/4}\frac{1}{\sqrt{2}}|-\rangle$$

b) Inner products

$$\begin{aligned}
 \langle \psi_1 | \psi_1 \rangle &= \left(\frac{1}{\sqrt{3}} \langle + | - i \frac{\sqrt{2}}{\sqrt{3}} \langle - | \right) \left(\frac{1}{\sqrt{3}} | + \rangle + i \frac{\sqrt{2}}{\sqrt{3}} | - \rangle \right) = \frac{1}{3} + \frac{2}{3} = 1 \\
 \langle \psi_1 | \psi_2 \rangle &= \left(\frac{1}{\sqrt{3}} \langle + | - i \frac{\sqrt{2}}{\sqrt{3}} \langle - | \right) \left(\frac{1}{\sqrt{5}} | + \rangle - \frac{2}{\sqrt{5}} | - \rangle \right) = \frac{1}{\sqrt{15}} + \frac{2\sqrt{2}i}{\sqrt{15}} = \frac{1}{\sqrt{15}} (1 + i2\sqrt{2}) \\
 \langle \psi_1 | \psi_3 \rangle &= \left(\frac{1}{\sqrt{3}} \langle + | - i \frac{\sqrt{2}}{\sqrt{3}} \langle - | \right) \left(\frac{1}{\sqrt{2}} | + \rangle + \frac{e^{i\pi/4}}{\sqrt{2}} | - \rangle \right) = \frac{1}{\sqrt{6}} - \frac{\sqrt{2}ie^{i\pi/4}}{\sqrt{6}} = \frac{1}{\sqrt{6}} (2 - i) \\
 \langle \psi_2 | \psi_1 \rangle &= \left(\frac{1}{\sqrt{5}} \langle + | - \frac{2}{\sqrt{5}} \langle - | \right) \left(\frac{1}{\sqrt{3}} | + \rangle + i \frac{\sqrt{2}}{\sqrt{3}} | - \rangle \right) = \frac{1}{\sqrt{15}} - \frac{2i}{\sqrt{15}} = \frac{1}{\sqrt{15}} (1 - i2\sqrt{2}) \\
 \langle \psi_2 | \psi_2 \rangle &= \left(\frac{1}{\sqrt{5}} \langle + | - \frac{2}{\sqrt{5}} \langle - | \right) \left(\frac{1}{\sqrt{5}} | + \rangle - \frac{2}{\sqrt{5}} | - \rangle \right) = \frac{1}{5} + \frac{4}{5} = 1 \\
 \langle \psi_2 | \psi_3 \rangle &= \left(\frac{1}{\sqrt{5}} \langle + | - \frac{2}{\sqrt{5}} \langle - | \right) \left(\frac{1}{\sqrt{2}} | + \rangle + \frac{e^{i\pi/4}}{\sqrt{2}} | - \rangle \right) = \frac{1}{\sqrt{10}} - \frac{2e^{i\pi/4}}{\sqrt{10}} = \frac{1}{\sqrt{10}} (1 - \sqrt{2} - i\sqrt{2}) \\
 \langle \psi_3 | \psi_1 \rangle &= \left(\frac{1}{\sqrt{2}} \langle + | + \frac{e^{-i\pi/4}}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{3}} | + \rangle + i \frac{\sqrt{2}}{\sqrt{3}} | - \rangle \right) = \frac{1}{\sqrt{6}} + \frac{\sqrt{2}ie^{-i\pi/4}}{\sqrt{6}} = \frac{1}{\sqrt{6}} (2 + i) \\
 \langle \psi_3 | \psi_2 \rangle &= \left(\frac{1}{\sqrt{2}} \langle + | + \frac{e^{-i\pi/4}}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{5}} | + \rangle - \frac{2}{\sqrt{5}} | - \rangle \right) = \frac{1}{\sqrt{10}} - \frac{2e^{-i\pi/4}}{\sqrt{10}} = \frac{1}{\sqrt{10}} (1 - \sqrt{2} + i\sqrt{2}) \\
 \langle \psi_3 | \psi_3 \rangle &= \left(\frac{1}{\sqrt{2}} \langle + | + \frac{e^{-i\pi/4}}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{2}} | + \rangle + \frac{e^{i\pi/4}}{\sqrt{2}} | - \rangle \right) = \frac{1}{2} + \frac{1}{2} = 1
 \end{aligned}$$

1.3 Probability of measuring a_n in state $|\psi\rangle$ is

$$P_{a_n} = |\langle a_n | \psi \rangle|^2$$

Probability of same measurement if state is changed to $e^{i\delta}|\psi\rangle$ is

$$\begin{aligned}
 P_{a_n, NEW} &= |\langle a_n | e^{i\delta} \psi \rangle|^2 \\
 &= |e^{i\delta} \langle a_n | \psi \rangle|^2 \\
 &= |\langle a_n | \psi \rangle|^2
 \end{aligned}$$

So the probability is unchanged.

1.4

$$\begin{aligned}
 |+\rangle_x &= a|+\rangle + b|-\rangle & P_{1,-x} &= |{}_x\langle - | + \rangle|^2 = \frac{1}{2} \\
 |-\rangle_x &= c|+\rangle + d|-\rangle & P_{2,+x} &= |{}_x\langle + | - \rangle|^2 = \frac{1}{2} \\
 & & P_{2,-x} &= |{}_x\langle - | - \rangle|^2 = \frac{1}{2} \\
 P_{1,-x} &= |{}_x\langle - | + \rangle|^2 = |\{c^* \langle + | + d^* \langle - | \} | + \rangle|^2 = |c^*|^2 = |c|^2 \Rightarrow |c|^2 = \frac{1}{2} \\
 P_{2,+x} &= |{}_x\langle + | - \rangle|^2 = |\{a^* \langle + | + b^* \langle - | \} | - \rangle|^2 = |b^*|^2 = |b|^2 \Rightarrow |b|^2 = \frac{1}{2} \\
 P_{2,-x} &= |{}_x\langle - | - \rangle|^2 = |\{c^* \langle + | + d^* \langle - | \} | - \rangle|^2 = |d^*|^2 = |d|^2 \Rightarrow |d|^2 = \frac{1}{2}
 \end{aligned}$$

Ch. 1 Solutions

1.5 a) Possible results of a measurement of the spin component S_z are always $\pm \hbar/2$ for a spin-1/2 particle. Probabilities are

$$P_{+\hbar/2} = |\langle + | \psi \rangle|^2 = \left| \left\langle + \left| \left(\frac{2}{\sqrt{13}} |+\rangle + i \frac{3}{\sqrt{13}} |-\rangle \right) \right\rangle \right|^2 = \left| \frac{2}{\sqrt{13}} \right|^2 = \frac{4}{13}$$

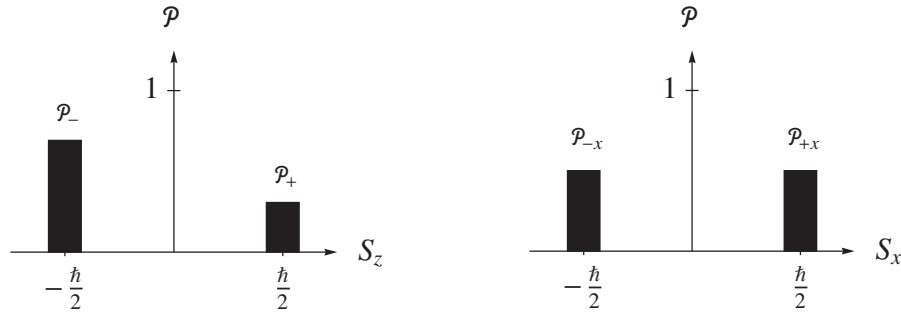
$$P_{-\hbar/2} = |\langle - | \psi \rangle|^2 = \left| \left\langle - \left| \left(\frac{2}{\sqrt{13}} |+\rangle + i \frac{3}{\sqrt{13}} |-\rangle \right) \right\rangle \right|^2 = \left| \frac{3i}{\sqrt{13}} \right|^2 = \frac{9}{13}$$

b) Possible results of a measurement of the spin component S_x are always $\pm \hbar/2$ for a spin-1/2 particle. Probabilities are

$$P_{+x} = \left| {}_x \langle + | \psi \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{2}{\sqrt{13}} |+\rangle + i \frac{3}{\sqrt{13}} |-\rangle \right) \right|^2 = \left| \frac{2}{\sqrt{26}} + i \frac{3}{\sqrt{26}} \right|^2 = \frac{1}{2}$$

$$P_{-x} = \left| {}_x \langle - | \psi \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{2}{\sqrt{13}} |+\rangle + i \frac{3}{\sqrt{13}} |-\rangle \right) \right|^2 = \left| \frac{2}{\sqrt{26}} - i \frac{3}{\sqrt{26}} \right|^2 = \frac{1}{2}$$

c) Histogram:



1.6 a) Possible results of a measurement of the spin component S_z are always $\pm \hbar/2$ for a spin-1/2 particle. Probabilities are

$$P_{+\hbar/2} = |\langle + | \psi \rangle|^2 = \left| \left\langle + \left| \left(\frac{2}{\sqrt{13}} |+\rangle_x + i \frac{3}{\sqrt{13}} |-\rangle_x \right) \right\rangle \right|^2 = \left| \frac{2}{\sqrt{26}} + i \frac{3}{\sqrt{26}} \right|^2 = \frac{1}{2}$$

$$P_{-\hbar/2} = |\langle - | \psi \rangle|^2 = \left| \left\langle - \left| \left(\frac{2}{\sqrt{13}} |+\rangle_x + i \frac{3}{\sqrt{13}} |-\rangle_x \right) \right\rangle \right|^2 = \left| \frac{2}{\sqrt{26}} - i \frac{3}{\sqrt{26}} \right|^2 = \frac{1}{2}$$

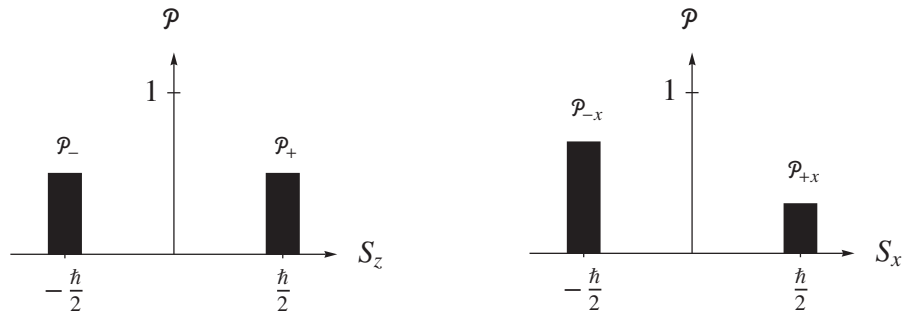
b) Possible results of a measurement of the spin component S_x are always $\pm \hbar/2$ for a spin-1/2 particle. Probabilities are

$$P_{+x} = \left| {}_x \langle + | \psi \rangle \right|^2 = \left| {}_x \langle + | \left(\frac{2}{\sqrt{13}} |+\rangle_x + i \frac{3}{\sqrt{13}} |-\rangle_x \right) \right|^2 = \left| \frac{2}{\sqrt{13}} \right|^2 = \frac{4}{13}$$

$$P_{-x} = \left| {}_x \langle - | \psi \rangle \right|^2 = \left| {}_x \langle - | \left(\frac{2}{\sqrt{13}} |+\rangle_x + i \frac{3}{\sqrt{13}} |-\rangle_x \right) \right|^2 = \left| \frac{3i}{\sqrt{13}} \right|^2 = \frac{9}{13}$$

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c) Histogram:



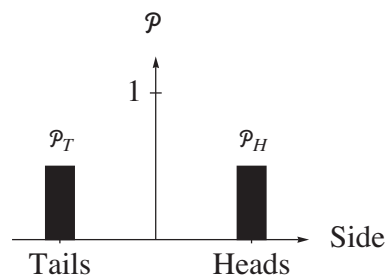
1.7 a) Heads or tails: H or T

b) Each result is equally likely so

$$P_H = \frac{1}{2}$$

$$P_T = \frac{1}{2}$$

c) Histogram:

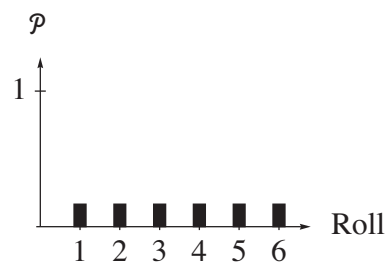


1.8 a) Six sides with 1, 2, 3, 4, 5, or 6 dots.

b) Each result is equally likely so

$$P_1 = P_2 = P_3 = P_4 = P_5 = P_6 = \frac{1}{6}$$

c) Histogram:



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1.9 a) 36 possible die combinations with 11 possible numerical results:

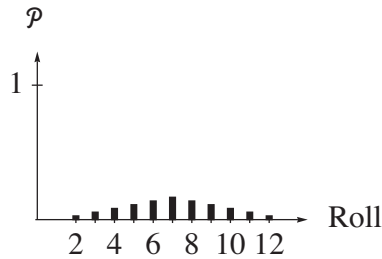
$$\begin{aligned}
 2 &= 1 + 1 \\
 3 &= 1 + 2, 2 + 1 \\
 4 &= 1 + 3, 2 + 2, 3 + 1 \\
 5 &= 1 + 4, 2 + 3, 3 + 2, 4 + 1 \\
 6 &= 1 + 5, 2 + 4, 3 + 3, 4 + 2, 5 + 1 \\
 7 &= 1 + 6, 2 + 5, 3 + 4, 4 + 3, 5 + 2, 6 + 1 \\
 8 &= 2 + 6, 3 + 5, 4 + 4, 5 + 3, 6 + 2 \\
 9 &= 3 + 6, 4 + 5, 5 + 4, 6 + 3 \\
 10 &= 4 + 6, 5 + 5, 6 + 4 \\
 11 &= 5 + 6, 6 + 5 \\
 12 &= 6 + 6
 \end{aligned}$$

b) Each possible die combination is equally likely, so the probabilities of the numerical results are the number of possible combinations divided by 36:

$$\begin{aligned}
 P_2 &= \frac{1}{36}, P_3 = \frac{2}{36} = \frac{1}{18}, P_4 = \frac{3}{36} = \frac{1}{12}, P_5 = \frac{4}{36} = \frac{1}{9}, P_6 = \frac{5}{36}, P_7 = \frac{6}{36} = \frac{1}{6}, \\
 P_8 &= \frac{5}{36}, P_9 = \frac{4}{36} = \frac{1}{9}, P_{10} = \frac{3}{36} = \frac{1}{12}, P_{11} = \frac{2}{36} = \frac{1}{18}, P_{12} = \frac{1}{36}
 \end{aligned}$$

Note that the sum of the probabilities is unity as it must be.

c) Histogram:



1.10 a) The probabilities for state 1 are

$$\begin{aligned}
 P_{1,+} &= \left| \langle + | \psi_1 \rangle \right|^2 = \left| \left\langle + \left| \left(\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| \frac{4}{5} \right|^2 = \frac{16}{25} \\
 P_{1,-} &= \left| \langle - | \psi_1 \rangle \right|^2 = \left| \left\langle - \left| \left(\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| i \frac{3}{5} \right|^2 = \frac{9}{25} \\
 P_{1,+x} &= \left| {}_x \langle + | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{4}{5} + \frac{i}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{1}{2} \\
 P_{1,-x} &= \left| {}_x \langle - | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{4}{5} - \frac{i}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{1}{2} \\
 P_{1,+y} &= \left| {}_y \langle + | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{4}{5} + \frac{1}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{49}{50} \\
 P_{1,-y} &= \left| {}_y \langle - | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{4}{5} - \frac{1}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{1}{50}
 \end{aligned}$$

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The probabilities for state 2 are

$$\begin{aligned}
 P_{2,+} &= \left| \langle + | \psi_2 \rangle \right|^2 = \left| \left\langle + \left| \left(\frac{4}{5} |+\rangle - i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| \frac{4}{5} \right|^2 = \frac{16}{25} \\
 P_{2,-} &= \left| \langle - | \psi_2 \rangle \right|^2 = \left| \left\langle - \left| \left(\frac{4}{5} |+\rangle - i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| -i \frac{3}{5} \right|^2 = \frac{9}{25} \\
 P_{2,+x} &= \left| \langle + | \psi_2 \rangle \right|^2 = \left| \left\langle + \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{4}{5} |+\rangle - i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{4}{5} - \frac{i}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{1}{2} \\
 P_{2,-x} &= \left| \langle - | \psi_2 \rangle \right|^2 = \left| \left\langle - \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{4}{5} |+\rangle - i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{4}{5} + \frac{i}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{1}{2} \\
 P_{2,+y} &= \left| \langle + | \psi_2 \rangle \right|^2 = \left| \left\langle + \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{4}{5} |+\rangle - i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{4}{5} - \frac{1}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{1}{50} \\
 P_{2,-y} &= \left| \langle - | \psi_2 \rangle \right|^2 = \left| \left\langle - \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{i}{\sqrt{2}} \langle - | \right) \left(\frac{4}{5} |+\rangle - i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \frac{4}{5} + \frac{1}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{49}{50}
 \end{aligned}$$

The probabilities for state 3 are

$$\begin{aligned}
 P_{3,+} &= \left| \langle + | \psi_3 \rangle \right|^2 = \left| \left\langle + \left| \left(-\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| -\frac{4}{5} \right|^2 = \frac{16}{25} \\
 P_{3,-} &= \left| \langle - | \psi_3 \rangle \right|^2 = \left| \left\langle - \left| \left(-\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| i \frac{3}{5} \right|^2 = \frac{9}{25} \\
 P_{3,+x} &= \left| \langle + | \psi_3 \rangle \right|^2 = \left| \left\langle + \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) \left(-\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| -\frac{1}{\sqrt{2}} \frac{4}{5} + \frac{i}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{1}{2} \\
 P_{3,-x} &= \left| \langle - | \psi_3 \rangle \right|^2 = \left| \left\langle - \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{1}{\sqrt{2}} \langle - | \right) \left(-\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| -\frac{1}{\sqrt{2}} \frac{4}{5} - \frac{i}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{1}{2} \\
 P_{3,+y} &= \left| \langle + | \psi_3 \rangle \right|^2 = \left| \left\langle + \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{i}{\sqrt{2}} \langle - | \right) \left(-\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| -\frac{1}{\sqrt{2}} \frac{4}{5} + \frac{1}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{1}{50} \\
 P_{3,-y} &= \left| \langle - | \psi_3 \rangle \right|^2 = \left| \left\langle - \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{i}{\sqrt{2}} \langle - | \right) \left(-\frac{4}{5} |+\rangle + i \frac{3}{5} |-\rangle \right) \right\rangle \right|^2 = \left| -\frac{1}{\sqrt{2}} \frac{4}{5} - \frac{1}{\sqrt{2}} \frac{3}{5} \right|^2 = \frac{49}{50}
 \end{aligned}$$

b) States 2 and 3 differ only by an overall phase of $e^{i\pi} = -1$, so the measurement results are the same; the states are physically indistinguishable. States 1 and 2 have different relative phases between the coefficients, so they produce different results.

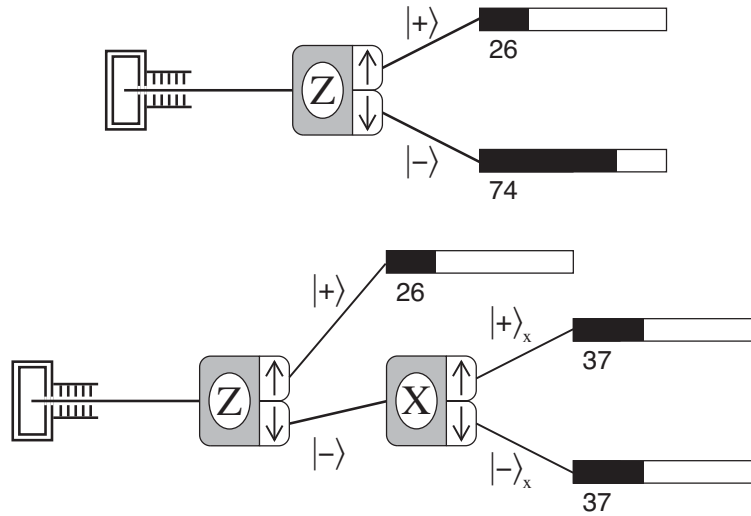
1.11 a) Possible results of a measurement of the spin component S_z are always $\pm \hbar/2$ for a spin-1/2 particle. Probabilities are

$$\begin{aligned}
 P_{+\hbar/2} &= \left| \langle + | \psi \rangle \right|^2 = \left| \left\langle + \left| \left(\frac{3}{\sqrt{34}} |+\rangle + i \frac{5}{\sqrt{34}} |-\rangle \right) \right\rangle \right|^2 = \left| \frac{3}{\sqrt{34}} \right|^2 = \frac{9}{34} \cong 0.26 \\
 P_{-\hbar/2} &= \left| \langle - | \psi \rangle \right|^2 = \left| \left\langle - \left| \left(\frac{3}{\sqrt{34}} |+\rangle + i \frac{5}{\sqrt{34}} |-\rangle \right) \right\rangle \right|^2 = \left| i \frac{5}{\sqrt{34}} \right|^2 = \frac{25}{34} \cong 0.74
 \end{aligned}$$

b) After the measurement result of the spin component S_z is $-\hbar/2$, the system is in the $|-\rangle$ eigenstate corresponding to that result. The possible results of a measurement of the spin component S_x are always $\pm \hbar/2$ for a spin-1/2 particle. The probabilities are

$$\begin{aligned}
 P_{+x} &= \left| \langle + | \psi_{after} \rangle \right|^2 = \left| \langle + | - \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) | - \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2} \\
 P_{-x} &= \left| \langle - | \psi_{after} \rangle \right|^2 = \left| \langle - | - \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \frac{1}{\sqrt{2}} \langle - | \right) | - \rangle \right|^2 = \left| -\frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}
 \end{aligned}$$

c) Diagrams



1.12 For a system with three possible measurement results: a_1 , a_2 , and a_3 , the three eigenstates are $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$

Orthogonality:

$$\langle a_1 | a_2 \rangle = 0$$

$$\langle a_1 | a_3 \rangle = 0$$

$$\langle a_2 | a_3 \rangle = 0$$

Normalization:

$$\langle a_1 | a_1 \rangle = 1$$

$$\langle a_2 | a_2 \rangle = 1$$

$$\langle a_3 | a_3 \rangle = 1$$

Completeness:

$$|\psi\rangle = c_1 |a_1\rangle + c_2 |a_2\rangle + c_3 |a_3\rangle$$

1.13 a) For a system with three possible measurement results: a_1 , a_2 , and a_3 , the three eigenstates $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$ are

$$|a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |a_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |a_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Ch. 1 Solutions

b) In matrix notation, the state is

$$|\psi\rangle = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

The state given is not normalized, so first we normalize it:

$$|\psi\rangle = C \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$1 = \langle\psi|\psi\rangle = C^* \begin{pmatrix} 1 & -2 & 5 \end{pmatrix} C \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = C^* C (1 + 4 + 25) = 1 \Rightarrow C = 1/\sqrt{30}$$

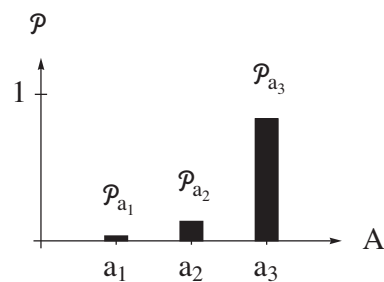
The probabilities are

$$P_{a_1} = |\langle a_1 | \psi \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{30}} \right|^2 = \frac{1}{30}$$

$$P_{a_2} = |\langle a_2 | \psi \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \right|^2 = \left| -\frac{2}{\sqrt{30}} \right|^2 = \frac{4}{30}$$

$$P_{a_3} = |\langle a_3 | \psi \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \right|^2 = \left| \frac{5}{\sqrt{30}} \right|^2 = \frac{25}{30}$$

Histogram:



c) In matrix notation, the state is

$$|\psi\rangle = \begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}$$

Ch. 1 Solutions

The state given is not normalized, so first we normalize it:

$$|\psi\rangle = C \begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}$$

$$\langle\psi|\psi\rangle = C^* \begin{pmatrix} 2 & -3i & 0 \end{pmatrix} C \begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix} = C^* C (4 + 9 + 0) = 1 \Rightarrow C = 1/\sqrt{13}$$

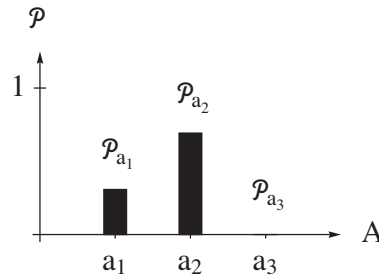
The probabilities are

$$P_{a_1} = |\langle a_1 | \psi \rangle|^2 = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix} \right|^2 = \left| \frac{2}{\sqrt{13}} \right|^2 = \frac{4}{13}$$

$$P_{a_2} = |\langle a_2 | \psi \rangle|^2 = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix} \right|^2 = \left| \frac{3i}{\sqrt{13}} \right|^2 = \frac{9}{13}$$

$$P_{a_3} = |\langle a_3 | \psi \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix} \right|^2 = \left| \frac{0}{\sqrt{13}} \right|^2 = 0$$

Histogram:



1.14. There are four possible measurement results: 2 eV, 4 eV, 7 eV, and 9 eV. The probabilities are

$$P_{2 \text{ eV}} = |\langle 2 \text{ eV} | \psi \rangle|^2 = \left| \langle 2 \text{ eV} | \frac{1}{\sqrt{39}} \{ 3|2 \text{ eV}\rangle - i|4 \text{ eV}\rangle + 2e^{i\pi/7}|7 \text{ eV}\rangle + 5|9 \text{ eV}\rangle \} \right|^2 = \frac{9}{39}$$

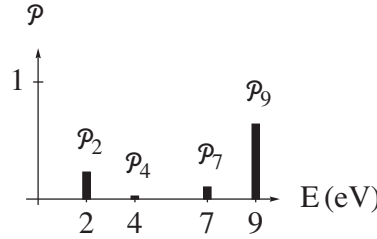
$$P_{4 \text{ eV}} = |\langle 4 \text{ eV} | \psi \rangle|^2 = \left| \langle 4 \text{ eV} | \frac{1}{\sqrt{39}} \{ 3|2 \text{ eV}\rangle - i|4 \text{ eV}\rangle + 2e^{i\pi/7}|7 \text{ eV}\rangle + 5|9 \text{ eV}\rangle \} \right|^2 = \frac{1}{39}$$

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$$P_{7 \text{ eV}} = \left| \langle 7 \text{ eV} | \psi \rangle \right|^2 = \left| \langle 2 \text{ eV} | \frac{1}{\sqrt{39}} \{ 3|2 \text{ eV}\rangle - i|4 \text{ eV}\rangle + 2e^{i\pi/7}|7 \text{ eV}\rangle + 5|9 \text{ eV}\rangle \} \right|^2 = \frac{4}{39}$$

$$P_{9 \text{ eV}} = \left| \langle 9 \text{ eV} | \psi \rangle \right|^2 = \left| \langle 2 \text{ eV} | \frac{1}{\sqrt{39}} \{ 3|2 \text{ eV}\rangle - i|4 \text{ eV}\rangle + 2e^{i\pi/7}|7 \text{ eV}\rangle + 5|9 \text{ eV}\rangle \} \right|^2 = \frac{25}{39}$$

Histogram:



1.15 The probability is

$$\begin{aligned} P_{\psi_f} &= \left| \langle \psi_f | \psi_i \rangle \right|^2 = \left| \left(\frac{1-i}{\sqrt{3}} \langle a_1 | + \frac{1}{\sqrt{6}} \langle a_2 | + \frac{1}{\sqrt{6}} \langle a_3 | \right) \left(\frac{i}{\sqrt{3}} |a_1\rangle + \sqrt{\frac{2}{3}} |a_2\rangle \right) \right|^2 \\ &= \left| \frac{i-i}{\sqrt{3}} + \sqrt{\frac{2}{3}} \frac{1}{\sqrt{6}} \right|^2 = \left| \frac{i}{3} + \frac{1}{3} + \frac{1}{3} \right|^2 = \frac{1}{9} + \frac{4}{9} = \frac{5}{9} \end{aligned}$$

1.16 The measured probabilities are

$$\begin{aligned} P_+ &= \frac{1}{2} & P_{+x} &= \frac{3}{4} & P_{+y} &= 0.067 \\ P_- &= \frac{1}{2} & P_{-x} &= \frac{1}{4} & P_{-y} &= 0.933 \end{aligned}$$

Write the input state as

$$|\psi\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned} P_+ &= \left| \langle + | \psi \rangle \right|^2 = \left| \langle + | \{ a|+\rangle + b|-\rangle \} \right|^2 = |a|^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}} \\ P_- &= \left| \langle - | \psi \rangle \right|^2 = \left| \langle - | \{ a|+\rangle + b|-\rangle \} \right|^2 = |b|^2 = \frac{1}{2} \Rightarrow b = \frac{1}{\sqrt{2}} e^{i\phi} \end{aligned}$$

allowing for a possible relative phase. Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned} P_{+x} &= \left| \langle +_x | \psi \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \{ \langle + | + \langle - | \} \frac{1}{\sqrt{2}} \{ |+\rangle + e^{i\phi} |-\rangle \} \right|^2 = \left| \frac{1}{2} \{ 1 + e^{i\phi} \} \right|^2 \\ &= \frac{1}{4} \{ 1 + e^{i\phi} \} \{ 1 + e^{-i\phi} \} = \frac{1}{4} \{ 1 + 1 + e^{i\phi} + e^{-i\phi} \} = \frac{1}{2} \{ 1 + \cos \phi \} = \frac{3}{4} \\ \cos \phi &= \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \end{aligned}$$

Ch. 1 Solutions

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned} P_{+y} &= \left| \langle + | \psi \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \left\{ \langle + | - \rangle \frac{1}{\sqrt{2}} \left\{ | + \rangle + e^{i\phi} | - \rangle \right\} \right\} \right|^2 = \left| \frac{1}{2} \{ 1 - ie^{i\phi} \} \right|^2 \\ &= \frac{1}{4} \{ 1 - ie^{i\phi} \} \{ 1 + ie^{-i\phi} \} = \frac{1}{4} \{ 1 + 1 - ie^{i\phi} + ie^{-i\phi} \} = \frac{1}{2} \{ 1 + \sin \phi \} = 0.067 \\ \sin \phi &= -0.866 \Rightarrow \phi = \frac{4\pi}{3} \text{ or } \frac{5\pi}{3} \Rightarrow \phi = \frac{5\pi}{3} \end{aligned}$$

Hence the input state is

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle + e^{i\frac{5\pi}{3}} |-\rangle \right) = |+\rangle_{\hat{n}(\theta=\frac{\pi}{2}, \phi=\frac{5\pi}{3})}$$

1.17 Follow the solution method given in the lab handout. (i) For unknown number 1, the measured probabilities are

$$\begin{aligned} P_+ &= 1 & P_{+x} &= \frac{1}{2} & P_{+y} &= \frac{1}{2} \\ P_- &= 0 & P_{-x} &= \frac{1}{2} & P_{-y} &= \frac{1}{2} \end{aligned}$$

Write the unknown state as

$$|\psi_1\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned} P_+ &= \left| \langle + | \psi_1 \rangle \right|^2 = \left| \langle + | \{ a|+\rangle + b|-\rangle \} \right|^2 = |a|^2 = 1 \Rightarrow a = 1 \\ P_- &= \left| \langle - | \psi_1 \rangle \right|^2 = \left| \langle - | \{ a|+\rangle + b|-\rangle \} \right|^2 = |b|^2 = 0 \Rightarrow b = 0 \end{aligned}$$

Hence the unknown state is

$$|\psi_1\rangle = |+\rangle$$

which produces the probabilities

$$\begin{aligned} P_+ &= \left| \langle + | \psi_1 \rangle \right|^2 = \left| \langle + | + \rangle \right|^2 = 1 \\ P_- &= \left| \langle - | \psi_1 \rangle \right|^2 = \left| \langle - | + \rangle \right|^2 = 0 \\ P_{+x} &= \left| \langle + | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \rangle + \frac{1}{\sqrt{2}} \langle - | + \rangle \right) \right|^2 = \frac{1}{2} \\ P_{-x} &= \left| \langle - | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | - \rangle + \frac{1}{\sqrt{2}} \langle - | - \rangle \right) \right|^2 = \frac{1}{2} \\ P_{+y} &= \left| \langle + | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \rangle - \frac{i}{\sqrt{2}} \langle - | + \rangle \right) \right|^2 = \frac{1}{2} \\ P_{-y} &= \left| \langle - | \psi_1 \rangle \right|^2 = \left| \left(\frac{1}{\sqrt{2}} \langle + | + \rangle + \frac{i}{\sqrt{2}} \langle - | + \rangle \right) \right|^2 = \frac{1}{2} \end{aligned}$$

in agreement with the experiment.

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(ii) For unknown number 2, the measured probabilities are

$$\begin{aligned} P_+ &= \frac{1}{2} & P_{+x} &= \frac{1}{2} & P_{+y} &= 0 \\ P_- &= \frac{1}{2} & P_{-x} &= \frac{1}{2} & P_{-y} &= 1 \end{aligned}$$

Write the unknown state as

$$|\psi_2\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned} P_+ &= |\langle + | \psi_2 \rangle|^2 = |\langle + | \{a|+\rangle + b|-\rangle\}|^2 = |a|^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}} \\ P_- &= |\langle - | \psi_2 \rangle|^2 = |\langle - | \{a|+\rangle + b|-\rangle\}|^2 = |b|^2 = \frac{1}{2} \Rightarrow b = \frac{1}{\sqrt{2}} e^{i\phi} \end{aligned}$$

allowing for a possible relative phase. Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned} P_{+x} &= \left| \langle + | \psi_2 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \left\{ \langle + | + \langle - | \right\} \frac{1}{\sqrt{2}} \left\{ |+\rangle + e^{i\phi} |-\rangle \right\} \right|^2 = \left| \frac{1}{2} \{1 + e^{i\phi}\} \right|^2 \\ &= \frac{1}{4} \{1 + e^{i\phi}\} \{1 + e^{-i\phi}\} = \frac{1}{4} \{1 + 1 + e^{i\phi} + e^{-i\phi}\} = \frac{1}{2} \{1 + \cos \phi\} = \frac{1}{2} \\ \cos \phi &= 0 \Rightarrow \phi = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \end{aligned}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned} P_{+y} &= \left| \langle + | \psi_2 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \left\{ \langle + | - i \langle - | \right\} \frac{1}{\sqrt{2}} \left\{ |+\rangle + e^{i\phi} |-\rangle \right\} \right|^2 = \left| \frac{1}{2} \{1 - ie^{i\phi}\} \right|^2 \\ &= \frac{1}{4} \{1 - ie^{i\phi}\} \{1 + ie^{-i\phi}\} = \frac{1}{4} \{1 + 1 - ie^{i\phi} + ie^{-i\phi}\} = \frac{1}{2} \{1 + \sin \phi\} = 0 \\ \sin \phi &= -1 \Rightarrow \phi = \frac{3\pi}{2} \end{aligned}$$

Hence the unknown state is

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle + e^{i\frac{3\pi}{2}} |-\rangle \right) = \frac{1}{\sqrt{2}} (|+\rangle - i|-\rangle) = |-\rangle_y$$

which produces the probabilities

$$\begin{aligned} P_+ &= |\langle + | \psi_2 \rangle|^2 = \left| \langle + | \frac{1}{\sqrt{2}} (|+\rangle - i|-\rangle) \right|^2 = \frac{1}{2} \\ P_- &= |\langle - | \psi_2 \rangle|^2 = \left| \langle - | \frac{1}{\sqrt{2}} (|+\rangle - i|-\rangle) \right|^2 = \frac{1}{2} \\ P_{+x} &= \left| \langle + | \psi_2 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | + \langle - |) \frac{1}{\sqrt{2}} (|+\rangle - i|-\rangle) \right|^2 = \frac{1}{2} \\ P_{-x} &= \left| \langle - | \psi_2 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | - \langle - |) \frac{1}{\sqrt{2}} (|+\rangle - i|-\rangle) \right|^2 = \frac{1}{2} \\ P_{+y} &= \left| \langle + | \psi_2 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | - i \langle - |) \frac{1}{\sqrt{2}} (|+\rangle - i|-\rangle) \right|^2 = 0 \\ P_{-y} &= \left| \langle - | \psi_2 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | + i \langle - |) \frac{1}{\sqrt{2}} (|+\rangle - i|-\rangle) \right|^2 = 1 \end{aligned}$$

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in agreement with the experiment.

(iii) For unknown number 3, the measured probabilities are

$$\begin{aligned} P_+ &= \frac{1}{2} & P_{+x} &= \frac{1}{4} & P_{+y} &= 0.067 \\ P_- &= \frac{1}{2} & P_{-x} &= \frac{3}{4} & P_{-y} &= 0.933 \end{aligned}$$

Write the unknown state as

$$|\psi_3\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned} P_+ &= |\langle + | \psi_3 \rangle|^2 = |\langle + | \{a|+\rangle + b|-\rangle\}|^2 = |a|^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}} \\ P_- &= |\langle - | \psi_3 \rangle|^2 = |\langle - | \{a|+\rangle + b|-\rangle\}|^2 = |b|^2 = \frac{1}{2} \Rightarrow b = \frac{1}{\sqrt{2}} e^{i\phi} \end{aligned}$$

allowing for a possible relative phase. Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned} P_{+x} &= |\langle +_x | \psi_3 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \left\{ \langle + | + \rangle \frac{1}{\sqrt{2}} \{ |+\rangle + e^{i\phi} |-\rangle \} \right\} \right|^2 = \left| \frac{1}{2} \{ 1 + e^{i\phi} \} \right|^2 \\ &= \frac{1}{4} \{ 1 + e^{i\phi} \} \{ 1 + e^{-i\phi} \} = \frac{1}{4} \{ 1 + 1 + e^{i\phi} + e^{-i\phi} \} = \frac{1}{2} \{ 1 + \cos \phi \} = \frac{1}{4} \\ \cos \phi &= -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \end{aligned}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned} P_{+y} &= |\langle +_y | \psi_3 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \left\{ \langle + | -i \rangle \frac{1}{\sqrt{2}} \{ |+\rangle + e^{i\phi} |-\rangle \} \right\} \right|^2 = \left| \frac{1}{2} \{ 1 - ie^{i\phi} \} \right|^2 \\ &= \frac{1}{4} \{ 1 - ie^{i\phi} \} \{ 1 + ie^{-i\phi} \} = \frac{1}{4} \{ 1 + 1 - ie^{i\phi} + ie^{-i\phi} \} = \frac{1}{2} \{ 1 + \sin \phi \} = 0.067 \\ \sin \phi &= -0.866 \Rightarrow \phi = \frac{4\pi}{3} \text{ or } \frac{5\pi}{3} \Rightarrow \phi = \frac{4\pi}{3} \end{aligned}$$

Hence the unknown state is

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle + e^{i\frac{4\pi}{3}} |-\rangle \right) = |+\rangle_{\hat{n}(\theta=\frac{\pi}{2}, \phi=\frac{4\pi}{3})}$$

which produces the probabilities

$$\begin{aligned} P_+ &= |\langle + | \psi_3 \rangle|^2 = \left| \langle + | \frac{1}{\sqrt{2}} \left(|+\rangle + e^{i\frac{4\pi}{3}} |-\rangle \right) \right|^2 = \frac{1}{2} \\ P_- &= |\langle - | \psi_3 \rangle|^2 = \left| \langle - | \frac{1}{\sqrt{2}} \left(|+\rangle + e^{i\frac{4\pi}{3}} |-\rangle \right) \right|^2 = \frac{1}{2} \\ P_{+x} &= |\langle +_x | \psi_3 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \left(\langle + | + \rangle \frac{1}{\sqrt{2}} \left(|+\rangle + e^{i\frac{4\pi}{3}} |-\rangle \right) \right) \right|^2 = \frac{1}{2} (1 + \cos \frac{4\pi}{3}) = \frac{1}{4} \\ P_{-x} &= |\langle -_x | \psi_3 \rangle|^2 = \left| \frac{1}{\sqrt{2}} \left(\langle + | - \rangle \frac{1}{\sqrt{2}} \left(|+\rangle + e^{i\frac{4\pi}{3}} |-\rangle \right) \right) \right|^2 = \frac{1}{2} (1 - \cos \frac{4\pi}{3}) = \frac{3}{4} \end{aligned}$$

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$$P_{+y} = \left| {}_y\langle + | \psi_3 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \left(\langle + | - i \langle - | \right) \frac{1}{\sqrt{2}} \left(| + \rangle + e^{i\frac{4\pi}{3}} | - \rangle \right) \right|^2 = \frac{1}{2} \left(1 + \sin \frac{4\pi}{3} \right) = \frac{1}{2} \left(1 - \frac{\sqrt{3}}{2} \right) = 0.067$$

$$P_{-y} = \left| {}_y\langle - | \psi_3 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \left(\langle + | + i \langle - | \right) \frac{1}{\sqrt{2}} \left(| + \rangle + e^{i\frac{4\pi}{3}} | - \rangle \right) \right|^2 = \frac{1}{2} \left(1 - \sin \frac{4\pi}{3} \right) = \frac{1}{2} \left(1 + \frac{\sqrt{3}}{2} \right) = 0.933$$

in agreement with the experiment.

(iv) For unknown number 4, the measured probabilities are

$$\begin{aligned} P_+ &= \frac{1}{4} & P_{+x} &= \frac{7}{8} & P_{+y} &= 0.283 \\ P_- &= \frac{3}{4} & P_{-x} &= \frac{1}{8} & P_{-y} &= 0.717 \end{aligned}$$

Write the unknown state as

$$|\psi_4\rangle = a|+\rangle + b|-\rangle$$

Equating the predicted S_z probabilities and the experimental results gives

$$\begin{aligned} P_+ &= \left| \langle + | \psi_4 \rangle \right|^2 = \left| \langle + | \{ a|+\rangle + b|-\rangle \} \right|^2 = |a|^2 = \frac{1}{4} \Rightarrow a = \frac{1}{2} \\ P_- &= \left| \langle - | \psi_4 \rangle \right|^2 = \left| \langle - | \{ a|+\rangle + b|-\rangle \} \right|^2 = |b|^2 = \frac{3}{4} \Rightarrow b = \frac{\sqrt{3}}{2} e^{i\phi} \end{aligned}$$

allowing for a possible relative phase. Equating the predicted S_x probabilities and the experimental results gives

$$\begin{aligned} P_{+x} &= \left| {}_x\langle + | \psi_4 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \left\{ \langle + | + \langle - | \right\} \frac{1}{2} \left\{ | + \rangle + \sqrt{3} e^{i\phi} | - \rangle \right\} \right|^2 = \left| \frac{1}{2\sqrt{2}} \left\{ 1 + \sqrt{3} e^{i\phi} \right\} \right|^2 \\ &= \frac{1}{8} \left\{ 1 + \sqrt{3} e^{i\phi} \right\} \left\{ 1 + \sqrt{3} e^{-i\phi} \right\} = \frac{1}{8} \left\{ 1 + 3 + \sqrt{3} e^{i\phi} + \sqrt{3} e^{-i\phi} \right\} = \frac{1}{4} \left\{ 2 + \sqrt{3} \cos \phi \right\} = \frac{7}{8} \\ \cos \phi &= \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6} \text{ or } \frac{11\pi}{6} \end{aligned}$$

Equating the predicted S_y probabilities and the experimental results gives

$$\begin{aligned} P_{+y} &= \left| {}_y\langle + | \psi_4 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \left\{ \langle + | - i \langle - | \right\} \frac{1}{2} \left\{ | + \rangle + \sqrt{3} e^{i\phi} | - \rangle \right\} \right|^2 = \left| \frac{1}{2\sqrt{2}} \left\{ 1 - i\sqrt{3} e^{i\phi} \right\} \right|^2 \\ &= \frac{1}{8} \left\{ 1 - i\sqrt{3} e^{i\phi} \right\} \left\{ 1 + i\sqrt{3} e^{-i\phi} \right\} = \frac{1}{8} \left\{ 1 + 3 - i\sqrt{3} e^{i\phi} + i\sqrt{3} e^{-i\phi} \right\} = \frac{1}{4} \left\{ 2 + \sqrt{3} \sin \phi \right\} = 0.283 \\ \sin \phi &= -0.50 \Rightarrow \phi = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \Rightarrow \phi = \frac{11\pi}{6} \end{aligned}$$

Hence the unknown state is

$$|\psi_4\rangle = \frac{1}{2}|+\rangle + \frac{\sqrt{3}}{2} e^{i\frac{11\pi}{6}} |-\rangle = \cos \frac{\pi}{3} |+\rangle + \sin \frac{\pi}{3} e^{i\frac{11\pi}{6}} |-\rangle = |+\rangle_{\hat{n}(\theta=\frac{2\pi}{3}, \phi=\frac{11\pi}{6})}$$

which produces the probabilities

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$$P_+ = \left| \langle + | \psi_4 \rangle \right|^2 = \left| \langle + | \frac{1}{2} (| + \rangle + \sqrt{3} e^{i \frac{11\pi}{6}} | - \rangle) \right|^2 = \frac{1}{4}$$

$$P_- = \left| \langle - | \psi_4 \rangle \right|^2 = \left| \langle - | \frac{1}{2} (| + \rangle + \sqrt{3} e^{i \frac{11\pi}{6}} | - \rangle) \right|^2 = \frac{3}{4}$$

$$P_{+x} = \left| {}_x \langle + | \psi_4 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | + \langle - |) \frac{1}{2} (| + \rangle + \sqrt{3} e^{i \frac{11\pi}{6}} | - \rangle) \right|^2 = \frac{1}{4} (2 + \sqrt{3} \cos \frac{11\pi}{6}) = \frac{7}{8}$$

$$P_{-x} = \left| {}_x \langle - | \psi_4 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | - \langle - |) \frac{1}{2} (| + \rangle + \sqrt{3} e^{i \frac{11\pi}{6}} | - \rangle) \right|^2 = \frac{1}{4} (2 - \sqrt{3} \cos \frac{11\pi}{6}) = \frac{1}{8}$$

$$P_{+y} = \left| {}_y \langle + | \psi_4 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | - i \langle - |) \frac{1}{2} (| + \rangle + \sqrt{3} e^{i \frac{11\pi}{6}} | - \rangle) \right|^2 = \frac{1}{4} (2 + \sqrt{3} \sin \frac{11\pi}{6}) = \frac{1}{4} (2 - \frac{\sqrt{3}}{2}) = 0.283$$

$$P_{-y} = \left| {}_y \langle - | \psi_4 \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} (\langle + | + i \langle - |) \frac{1}{2} (| + \rangle + \sqrt{3} e^{i \frac{11\pi}{6}} | - \rangle) \right|^2 = \frac{1}{4} (2 - \sqrt{3} \sin \frac{11\pi}{6}) = \frac{1}{4} (2 + \frac{\sqrt{3}}{2}) = 0.717$$

in agreement with the experiment.
