CHAPTER 3

Problem 3.1

From the given data,

$$\frac{k}{m} = \omega_n^2 = (2\pi f_n)^2 = (8\pi)^2 = 64\pi^2 \qquad \text{(a)}$$

$$\frac{k}{m + \Delta m} = (\omega_n')^2 = (2\pi f_n')^2 = (6\pi)^2 = 36\pi^2 \qquad \text{(b)}$$

Dividing Eq. (a) by Eq. (b) gives

$$1 + \frac{\Delta m}{m} = \frac{16}{9}$$

$$m = \frac{9}{7}\Delta m = \frac{9}{7}\frac{5}{\mathrm{g}} = \frac{6.43}{\mathrm{g}}\ \mathrm{lbs/g} \qquad \text{(c)}$$

From Eq. (a),

$$k = 64\pi^2 m = 64\pi^2\frac{6.43}{\mathrm{g}} = 10.52\ \mathrm{lbs/in.} \qquad \text{(d)}$$

Problem 3.2

At $\omega = \omega_n$, from Eq. (3.2.15),

$$u_o = (u_{st})_o \frac{1}{2\zeta} = 2 \qquad \text{(a)}$$

At $\omega = 0.1\omega_n$, from Eq. (3.2.13),

$$u_o \approx (u_{st})_o = 0.2$$

Substituting $(u_{st})_o = 0.2$ in Eq. (a) gives

$$\zeta = 0.05$$

Problem 3.3

Assuming that damping is small enough to justify the approximation that the resonant frequency is ω_n and the resonant amplitude of R_d is $1/2\zeta$, then the given data implies:

$$(u_o)_{\omega = \omega_n} = (u_{st})_o \frac{1}{2\zeta} \qquad \text{(a)}$$

$$(u_o)_{\omega = 1.2\omega_n} = (u_{st})_o \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}}$$

$$= (u_{st})_o \frac{1}{\sqrt{[1 - (1.2)^2]^2 + [2\zeta(1.2)]^2}} \qquad \text{(b)}$$

Combining Eq. (a) and Eq. (b):

$$\frac{1}{(2\zeta)^2}\left(\frac{(u_o)^2_{\omega=1.2\omega_n}}{(u_o)^2_{\omega=\omega_n}}\right)^2 = \frac{1}{(-0.44)^2 + (2.4\zeta)^2} \qquad \text{(c)}$$

For

$$\frac{(u_o)_{\omega = 1.2\omega_n}}{(u_o)_{\omega = \omega_n}} = \frac{1}{4}$$

Equation (c) gives

$$64\zeta^2 = 0.1935 + 5.76\zeta^2 \Rightarrow \zeta = 0.0576$$

Assumption of small damping implied in Eq. (a) is reasonable; otherwise we would have to use the exact resonant frequency $= \omega_n\sqrt{1 - 2\zeta^2}$ and exact resonant amplitude $= (u_{st})_o \Big/ \left[2\zeta\sqrt{1-2\zeta^2}\right]$.

Problem 3.4

(a) Machine running at 20 rpm.

$$\frac{\omega}{\omega_n} = \frac{20}{200} = 0.1$$

$$u_o = \frac{(u_{st})_o}{\left|1-(\omega/\omega_n)^2\right|} \tag{a}$$

or

$$0.2 = \frac{(u_{st})_o}{\left|1 - (0.1)^2\right|} \Rightarrow (u_{st})_o = 0.1980 \text{ in.}$$

For $\zeta = 0.25$ and $\omega/\omega_n = 0.1$,

$$u_o = (u_{st})_o \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \tag{b}$$

or

$$u_o = 0.1980 \frac{1}{\sqrt{[1 - (0.1)^2]^2 + [2(0.25)(0.1)]^2}}$$
$$= 0.1997 \text{ in.}$$

(b) Machine running at 180 rpm.

$$\frac{\omega}{\omega_n} = \frac{180}{200} = 0.9$$

From Eq. (a),

$$1.042 = \frac{(u_{st})_o}{\left|1 - (0.9)^2\right|} \Rightarrow (u_{st})_o = 0.1980 \text{ in.}$$

For $\zeta = 0.25$ and $\omega/\omega_n = 0.9$, Eq. (b) reads

$$u_o = 0.1980 \frac{1}{\sqrt{\left[1 - (0.9)^2\right]^2 + \left[2(0.25)(0.9)\right]^2}}$$
$$= 0.4053 \text{ in.}$$

(c) Machine running at 600 rpm.

$$\frac{\omega}{\omega_n} = \frac{600}{200} = 3$$

From Eq. (a),

$$0.0248 = \frac{(u_{st})_o}{\left|1 - (3)^2\right|} \Rightarrow (u_{st})_o = 0.1980 \text{ in.}$$

For $\zeta = 0.25$ and $\omega/\omega_n = 3$, Eq. (b) reads

$$u_o = 0.1980 \frac{1}{\sqrt{[1 - (3)^2]^2 + [2(0.25)(3)]^2}}$$
$$= 0.0243 \text{ in.}$$

(d) Summarizing these results together with given data:

ω/ω_n	$(u_o)_{\zeta = 0}$	$(u_o)_{\zeta = 0.25}$
0.1	0.2	0.1997
0.9	1.042	0.4053
3.0	0.0248	0.0243

The isolator is effective at $\omega/\omega_n = 0.9$; it reduces the deformation amplitude to 39% of the response without isolators. At $\omega/\omega_n = 0.1$ or 3, the isolator has essentially no influence on reducing the deformation.

Problem 3.5

Given:

$$w = 1200 \text{ lbs}, \quad E = 30 \times 10^6 \text{ psi},$$

$$I = 10 \text{ in.}^4, \quad L = 8 \text{ ft}; \quad \zeta = 1\%$$

$$p_o = 60 \text{ lbs}; \quad \omega = \left(\frac{300}{60}\right) 2\pi = 10\pi \text{rads/sec}$$

Stiffness of two beams:

$$k = 2\left(\frac{48EI}{L^3}\right) = 32{,}552 \text{ lbs/in.}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{w/g}} = \sqrt{\frac{32{,}552}{1200/386}} = 102.3 \text{ rads/sec}$$

Steady state response:

$$R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}}$$

where $\omega/\omega_n = 10\pi/102.3 = 0.3071$. Therefore,

$$R_d = \frac{1}{\sqrt{[1 - 0.0943]^2 + [2 \times 0.01 \times 0.3071]^2}} = 1.104$$

Displacement:

$$u_o = (u_{st})_o R_d = \frac{p_o}{k} R_d$$

$$= \frac{60}{32{,}552} \times 1.104 = 2.035 \times 10^{-3} \text{ in.}$$

Acceleration amplitude:

$$\ddot{u}_o = \omega^2 u_o = (10\pi)^2 \, 2.035 \times 10^{-3}$$

$$= 2.009 \text{ in./sec}^2 = 0.0052\text{g}$$

Problem 3.6

In Eq. (3.2.1) replacing the applied force by $p_o \cos\omega t$ and dividing by m we get

$$\ddot{u} + 2\zeta\omega_n \dot{u} + \omega_n^2 u = \frac{p_o}{m} \cos\omega t \qquad \text{(a)}$$

(a) The particular solution is of the form:

$$u_p(t) = C \sin\omega t + D \cos\omega t \qquad \text{(b)}$$

Differentiating once and then twice gives

$$\dot{u}_p(t) = C\omega \cos\omega t - D\omega \sin\omega t \qquad \text{(c)}$$

$$\ddot{u}_p(t) = -C\omega^2 \sin\omega t - D\omega^2 \cos\omega t \qquad \text{(d)}$$

Substituting Eqs. (b)-(d) in Eq. (a) and collecting terms:

$$\left[(\omega_n^2 - \omega^2)\,C - 2\zeta\omega_n\omega\,D\right] \sin\omega t$$

$$+ \left[2\zeta\omega_n\omega\,C + (\omega_n^2 - \omega^2)\,D\right] \cos\omega t = \frac{p_o}{m} \cos\omega t$$

Equating coefficients of $\sin\omega t$ and of $\cos\omega t$ on the two sides of the equation:

$$(\omega_n^2 - \omega^2)\,C - (2\zeta\omega_n\omega)\,D = 0 \qquad \text{(e)}$$

$$(2\zeta\omega_n\omega)\,C + (\omega_n^2 - \omega^2)\,D = \frac{p_o}{m} \qquad \text{(f)}$$

Solving Eqs. (e) and (f) for C and D gives

$$C = \frac{p_o}{m} \frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \qquad \text{(g)}$$

$$D = \frac{p_o}{m} \frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \qquad \text{(h)}$$

Substituting Eqs. (g) and (h) in Eq. (b) gives

$$u_p(t) = \frac{p_o}{k} \frac{\left[1 - (\omega/\omega_n)^2\right] \cos\omega t + \left[2\zeta\omega/\omega_n\right] \sin\omega t}{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}$$

(b) Maximum deformation is $u_o = \sqrt{C^2 + D^2}$. Substituting for C and D gives

$$u_o = \frac{p_o}{k} \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}}$$

This is same as Eq. (3.2.11) for the amplitude of deformation due to sinusoidal force.

Problem 3.7

(a) The displacement amplitude in given by Eq. (3.2.11):

$$u_o = (u_{st})_o \left[(1 - \beta^2)^2 + (2\zeta\beta)^2\right]^{-1/2} \qquad \text{(a)}$$

where $\beta = \omega/\omega_n$. Resonance occurs at β when u_o is maximum, i.e., $du_o/d\beta = 0$. Differentiating Eq. (a) with respect to β gives

$$-\frac{1}{2}\left[(1 - \beta^2)^2 + (2\zeta\beta)^2\right]^{-3/2} \times \left[2(1 - \beta^2)(-2\beta) + 2(2\zeta\beta)2\zeta\right] = 0$$

or

$$-4(1 - \beta^2)\beta + 8\zeta^2\beta = 0$$

or

$$1 - \beta^2 = 2\zeta^2 \Rightarrow \beta = \sqrt{1 - 2\zeta^2} \qquad \text{(b)}$$

Resonant frequency:

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

(b) Substituting Eq. (b) in Eq. (a) gives the resonant amplitude:

$$u_o = (u_{st})_o \frac{1}{\sqrt{(2\zeta^2)^2 + 4\zeta^2(1 - 2\zeta^2)}}$$

or

$$u_o = (u_{st})_o \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \qquad \text{(c)}$$

Problem 3.8

(a) From Eq. (3.2.19) the acceleration amplitude is

$$\ddot{u}_o = -\frac{p_o}{m} R_a = -\frac{p_o}{m}\left(\frac{\omega}{\omega_n}\right)^2 R_d$$

$$= -\frac{p_o}{m}\frac{\beta^2}{\left[(1-\beta^2)^2 + (2\zeta\beta)^2\right]^{1/2}} \qquad \text{(a)}$$

where $\beta = \omega/\omega_n$. Resonance occurs at β where $\ddot{u}_o$ is maximum, i.e., $d\ddot{u}_o/d\beta = 0$. Differentiating Eq. (a) with respect to β and setting the result equal to zero gives

$$\frac{1}{(1-\beta^2)^2+(2\zeta\beta)^2}\left\{2\beta\left[(1-\beta^2)^2 + (2\zeta\beta)^2\right]^{1/2} - \right.$$

$$\left.\left(\frac{\beta^2}{2}\right)\left[(1-\beta^2)^2 + (2\zeta\beta)^2\right]^{-1/2}\left[-4\beta(1-\beta^2) + 8\zeta^2\beta\right]\right\} = 0$$

Multiplying the numerator by $[(1-\beta^2)^2 + (2\zeta\beta)^2]^{1/2}$ and dividing it by β gives

$$2\left[(1-\beta^2)^2 + 4\zeta^2\beta^2\right] - \frac{\beta^2}{2}\left[-4(1-\beta^2) + 8\zeta^2\right] = 0$$

or

$$1 - 2\beta^2 + \beta^4 + 4\zeta^2\beta^2 + \beta^2 - \beta^4 - 2\zeta^2\beta^2 = 0$$

or

$$1 = \beta^2(1-2\zeta^2) \Rightarrow \beta = \frac{1}{\sqrt{1-2\zeta^2}} \qquad \text{(b)}$$

Resonant frequency:

$$\omega_r = \frac{\omega_n}{\sqrt{1-2\zeta^2}} \qquad \text{(c)}$$

(b) Dividing both the numerator and denominator of Eq. (a) by β^2 gives:

$$\ddot{u}_o = -\frac{p_o}{m}\frac{1}{\left[\left(\frac{1}{\beta^2}-1\right)^2 + \frac{4\zeta^2}{\beta^2}\right]^{1/2}} \qquad \text{(d)}$$

Substituting Eq. (b) in Eq. (d) gives the resonant amplitude:

$$\ddot{u}_o = -\frac{p_o}{m}\frac{1}{\left[(-2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)\right]^{1/2}}$$

$$\ddot{u}_o = -\frac{p_o}{m}\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Problem 3.9

(a) From Eq. (3.2.17), the velocity amplitude is

$$\dot{u}_0 = \frac{p_0}{\sqrt{km}} R_v = \frac{p_0}{\sqrt{km}} \left(\frac{\omega}{\omega_n} \right) R_d$$

$$= \frac{p_0}{\sqrt{km}} \frac{\beta}{[(1-\beta^2)^2 + (2\zeta\beta)^2]^{1/2}} \qquad \text{(a)}$$

where $\beta = \omega / \omega_n$. Resonance occurs at β where $\dot{u}$ is maximum, i.e., $d\dot{u}_0 / d\beta = 0$. Differentiating Eq. (a) with respect to β and setting the derivative equal to zero gives

$$[(1-\beta^2)^2 + (2\zeta\beta)^2]^{-1/2} -$$

$$\frac{\beta}{2}[(1-\beta^2)^2 + (2\zeta\beta)^2]^{-3/2}[2(1-\beta^2)(-2\beta) + 2(2\zeta\beta)2\zeta] = 0$$

Multiplying the numerator by $[(1-\beta^2)^2 + (2\zeta\beta)^2]^{3/2}$ gives

$$[(1-\beta^2)^2 + (2\zeta\beta)^2] - \frac{\beta}{2}[2(1-\beta^2)(-2\beta) + 2(2\zeta\beta)(2\zeta)] = 0$$

or

$$1 - 2\beta^2 + \beta^4 + 4\zeta^2\beta^2 - \beta(-2\beta + 2\beta^3 + 4\zeta^2\beta) = 0$$

or

$$\beta^4 = 1 \Rightarrow \beta = 1 \qquad \text{(b)}$$

Resonant frequency:

$$\omega_r = \omega_n \qquad \text{(c)}$$

(b) Substituding Eq. (b) into Eq. (a) gives

$$\dot{u}_0 = \frac{p_0}{\sqrt{km}} \frac{1}{2\zeta}$$

Problem 3.10

We assume that the structure has no mass other than the roof mass.

From Eq. (3.3.4),

$$\ddot{u}_o = \frac{m_e\, e\, \omega_n^2}{m} \left(\frac{\omega}{\omega_n} \right)^2 R_a \tag{a}$$

At $\omega = \omega_n$, $R_a = 1/2\zeta$ and Eq. (a) gives

$$\zeta = \frac{1}{2\ddot{u}_o} \frac{m_e}{m} e\, \omega_n^2 \tag{b}$$

Given:

$$m_e = 2 \times 50/\text{g} = 100 \text{ lbs/g} = 0.1 \text{ kips/g}$$

$$m = 500 \text{ kips/g}$$

$$e = 12 \text{ in.}$$

$$\omega_n = 2\pi f_n = 2\pi \times 4 = 8\pi$$

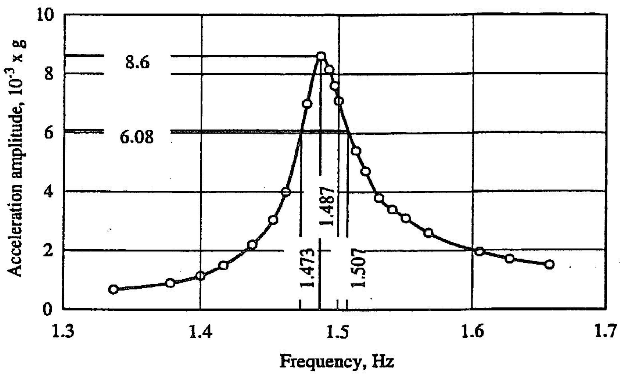
$$\ddot{u}_o = 0.02\text{g} = 7.72 \text{ in.}/\text{sec}^2$$

Substituting the above data in Eq. (b) gives

$$\zeta = \frac{1}{2\,(7.72)} \frac{0.1}{500} (12)\,(8\pi)^2 = 0.0982 = 9.82\%$$

Problem 3.11

The given data is plotted in the form of the frequency response curve shown in the accompanying figure:



(a) Natural frequency

The frequency response curve peaks at

$$f_n = 1.487 \text{ Hz}$$

Assuming small damping, this value is the natural frequency of the system.

(b) Damping ratio

The acceleration at the peak is $r_{peak} = 8.6 \times 10^{-3} \text{g}$. Draw a horizontal line at $r_{peak} \div \sqrt{2} = 6.08 \times 10^{-3}$ g to obtain f_a and f_b in Hz:

$$f_a = 1.473 \text{ Hz} \qquad f_b = 1.507 \text{ Hz}$$

Then,

$$\zeta = \frac{f_b - f_a}{2 f_n} = \frac{1.507 - 1.473}{2\,(1.487)} = 0.0114$$

$$= 1.14\%$$

Problem 3.12

(a) Transmissibility is given by Eq. (3.5.3) with $\zeta = 0$. Thus the force transmitted is

$$(f_{\mathbf{T}})_o = p_o \frac{1}{\left|1 - (\omega/\omega_n)^2\right|}$$

where $\omega_n = \sqrt{g/\delta_{st}}$ and $\omega = 2\pi f$. Therefore

$$(f_{\mathbf{T}})_o = p_o \frac{1}{\left|1 - \dfrac{4\pi^2}{\mathbf{g}} f^2 \delta_{st}\right|} \qquad \text{(a)}$$

(b) For $(f_{\mathrm{T}})_o = 0.1\, p_o$ and $f = 20$, Eq. (a) gives

$$0.1 = \frac{1}{\left|1 - (4\pi^2/386)(20)^2 \delta_{st}\right|} \quad \Rightarrow$$

$$1 - \frac{4\pi^2}{386}(20)^2 \delta_{st} = \pm 10 \quad \Rightarrow$$

$$\frac{4\pi^2}{386}(20)^2 \delta_{st} = -9 \text{ or } 11$$

The negative value is invalid. Therefore,

$$\delta_{st} = \frac{11}{(4\pi^2/386)(20)^2} = 0.269 \text{ in.}$$

Problem 3.13

The equation governing the deformation $u(t)$ in the suspension system is

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t) \qquad \text{(a)}$$

where $u_g(t) = u_{go} \sin \omega t$ and $\omega = 2\pi v/L$.

Substituting in Eq. (a) gives

$$m\ddot{u} + c\dot{u} + ku = m\omega^2 u_{go} \sin \omega t \qquad \text{(b)}$$

The amplitude of deformation is

$$u_o = (u_{st})_o R_d = \frac{m\omega^2 u_{go}}{k} R_d$$

The amplitude of the spring force is

$$f_{So} = ku_o = m\omega^2 u_{go} R_d \qquad \text{(c)}$$

where

$$R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(d)}$$

Numerical calculations:

$k = 800$ lbs/in.; $m = 4000/386$ lb-sec^2/in.

$u_{go} = 3$ in.

$\omega_n = 8.768$ rads/sec; $\omega = 3.686$ rads/sec

$\omega/\omega_n = 0.420$

Thus Eq. (d) gives

$$R_d = \frac{1}{\sqrt{[1 - (0.42)^2]^2 + [2(0.4)(0.42)]^2}} = 1.124$$

Substituting ω_n and R_d in Eq. (c) gives

$$f_{So} = \frac{4000}{386}(3.686)^2\, 3\,(1.124) = 474.8 \text{ lbs}$$

Problem 3.14

$$u_g(t) = u_{go} \sin \omega t \Rightarrow \ddot{u}_g(t) = -\omega^2 u_{go} \sin \omega t$$

$$p_{eff}(t) = -m\ddot{u}_g(t) = m\omega^2 u_{go} \sin \omega t \equiv p_o \sin \omega t$$

$$u_0 = (u_{st})_o R_d = \frac{m\omega^2 u_{go}}{k} R_d = u_{go} \left(\frac{\omega}{\omega_n} \right)^2 R_d = u_{go} R_a$$

Spring force: $f_S = ku_o$

The resonant frequency for f_S is the same as that for u_o, which is the resonsnat frequency for R_a (from section 3.2.5):

$$\omega_{res} = \frac{\omega_n}{\sqrt{1-2\zeta^2}}$$

$$= \frac{8.786}{\sqrt{1-2(0.4)^2}} = 10.655 \text{ rads / sec}$$

Resonance occurs at this forcing frequency, which implies a speed of

$$v = \frac{\omega L}{2\pi} = \frac{(10.655)100}{2\pi} = 169.6 \text{ ft / sec} = 116 \text{ mph}$$

Problem 3.15

Given: $w = 2000 \text{ lbs}$, $f = 1500 \text{ cpm} = 25 \text{ Hz}$, and $TR = 0.10$

For an undamped system,

$$TR = \frac{1}{\left|1 - (\omega/\omega_n)^2\right|} = 0.1$$

For $TR < 1$, $\omega/\omega_n > \sqrt{2}$ and the equation becomes

$$\frac{1}{(\omega/\omega_n)^2 - 1} = 0.1 \Rightarrow$$

$$(\omega/\omega_n)^2 = 11 \Rightarrow \omega/\omega_n = 3.32$$

$$\omega_n = \frac{\omega}{3.32} = \frac{2\pi f}{3.32} = \frac{2\pi(25)}{3.32} = 47.31 \text{ rads/sec}$$

$$k = \omega_n^2 m = \omega_n^2 w/\text{g} = (47.31)^2\ 2000/386$$
$$= 11.6 \text{ kips/in.}$$

Problem 3.16

The excitation is

$$u_g(t) = u_{go} \sin \omega t \quad \text{or} \quad \ddot{u}_g(t) = -\omega^2 u_{go} \sin \omega t \qquad \text{(a)}$$

The equation of motion is

$$m\ddot{u} + c\dot{u} + ku = p_{\text{eff}}(t) \qquad \text{(b)}$$

where

$$p_{\mathbf{eff}}(t) = -m\,\ddot{u}_g(t) = \omega^2 m\, u_{go} \sin \omega t \qquad \text{(c)}$$

The deformation response is given by Eq. (3.2.10) with p_o replaced by $\omega^2 m\, u_{go}$:

$$u(t) = \frac{\omega^2 m\, u_{go}}{k} R_d \sin(\omega t - \phi) \qquad \text{(d)}$$

where R_d and ϕ are defined by Eqs. (3.2.11) and (3.2.12), respectively.

The total displacement is

$$u^t(t) = u(t) + u_g(t)$$

Substituting Eqs. (a), (d), (3.2.11) and (3.2.12) gives

$$\begin{aligned} u^t(t) &= u_{go} \sin \omega_n t + \frac{(\omega/\omega_n)^2 u_{go}}{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\,\omega/\omega_n)^2} \\ &\quad \times \left\{\left[1-(\omega/\omega_n)^2\right]\sin \omega_n t - \left[2\zeta\,\omega/\omega_n\right]\cos \omega t\right\} \\ &= \frac{u_{go}}{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\,\omega/\omega_n)^2} \\ &\quad \times \left\{\left[1-(\omega/\omega_n)^2 + 4\zeta^2(\omega/\omega_n)^2\right] \sin \omega_n t - 2\zeta(\omega/\omega_n)^3 \cos \omega t\right\} \end{aligned} \qquad \text{(e)}$$

Equation (e) is of the form $u^t(t) = C \sin \omega t + D \cos \omega t$; hence $u_o^t = \sqrt{C^2 + D^2}$. Substituting for C and D gives

$$u_o^t = u_{go} \left\{ \frac{1 + (2\zeta\,\omega/\omega_n)^2}{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\,\omega/\omega_n\right]^2} \right\}^{1/2} \qquad \text{(f)}$$

Equation (f) is the same as Eq. (3.6.5).

Problem 3.17

The accelerometer properties are $f_n = 50$ cps and $\zeta = 0.7$; and the excitation is

$$\ddot{u}_g(t) = 0.1\text{g} \sin(2\pi f t);\ f = 10, 20 \text{ and } 40 \text{ Hz} \qquad \text{(a)}$$

Compare the excitation with the measured relative displacement

$$u(t) = \left[(-1/\omega_n^2)\, R_d\right] \ddot{u}_g(t - \phi/\omega) \qquad \text{(b)}$$

where

$$R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(c)}$$

and

$$\phi = \tan^{-1}\left[\frac{2\zeta\, \omega/\omega_n}{1 - (\omega/\omega_n)^2}\right] \qquad \text{(d)}$$

Because the instrument has been calibrated at low excitation frequencies, after separating the instrument constant $-1/\omega_n^2$, the recorded acceleration is

$$\ddot{u}_g(t) = R_d\, \ddot{u}_g(t - \phi/\omega) \qquad \text{(e)}$$

For a given f (or ω), ω/ω_n is computed and substituted in Eqs. (c) and (d) to calculate R_d and ϕ. With R_d and ϕ known for a given excitation frequency, the recorded acceleration is given by Eq. (e).

The computed amplitude R_d and time lag ϕ/ω agrees with Fig. 3.7.3. The difference between R_d and 1 is the error in measured acceleration (Table P3.17).

Table P3.17

f (Hz)	ω/ω_n	R_d	ϕ/ω, sec	% error in R_d
10	0.2	1	0.0045	0
20	0.4	0.991	0.0047	0.9
40	0.8	0.850	0.0050	15

Problem 3.18

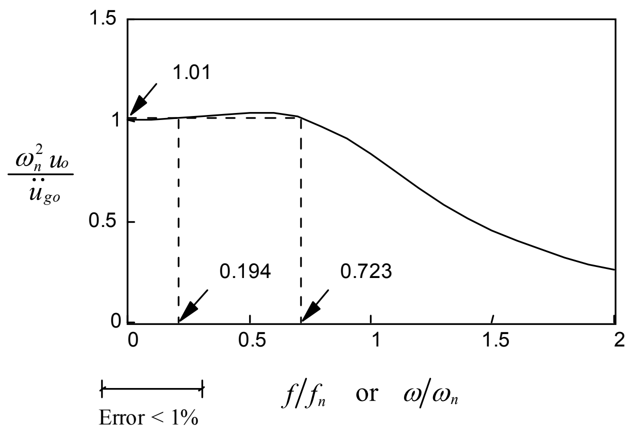
From Eq. (3.7.1),

$$\omega_n^2 u(t) = -R_d\, \ddot{u}_g(t - \phi/\omega)$$
$$= -R_d\, \ddot{u}_{go} \sin(2\pi f t - \phi)$$

and therefore,

$$\frac{\omega_n^2 u_o}{\ddot{u}_{go}} = R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \quad \text{(a)}$$

Equation (a) is plotted for $\zeta = 0.6$, the accelerometer damping ratio:



We want to bound R_d as follows:

$$0.99 \le R_d \le 1.01 \quad \text{(b)}$$

The relevant condition is $R_d \le 1.01$ because we are interested in a continuous range of frequencies over which the error is less than 1%. Therefore, impose

$$\frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} = 1.01 \quad \text{(c)}$$

Defining $\beta \equiv \omega/\omega_n$, Eq. (c) can be rewritten as

$$(1 - \beta^2)^2 + (2\zeta\beta)^2 = \left(\frac{1}{1.01}\right)^2 \Rightarrow$$

$$\beta^4 - 0.56\beta^2 + 1 = 0.9803 \Rightarrow$$

$$\beta^4 - 0.56\beta^2 + 0.0197 = 0 \Rightarrow$$

$$\beta^2 = 0.0377,\quad 0.5223 \Rightarrow \beta = 0.194,\quad 0.723$$

Choose $\beta = 0.194$ (see figure) which gives the desired frequency range:

$$f \le 0.194\, f_n = 0.194\,(25) \Rightarrow$$

$$f \le 4.86 \text{ Hz}$$

Problem 3.19

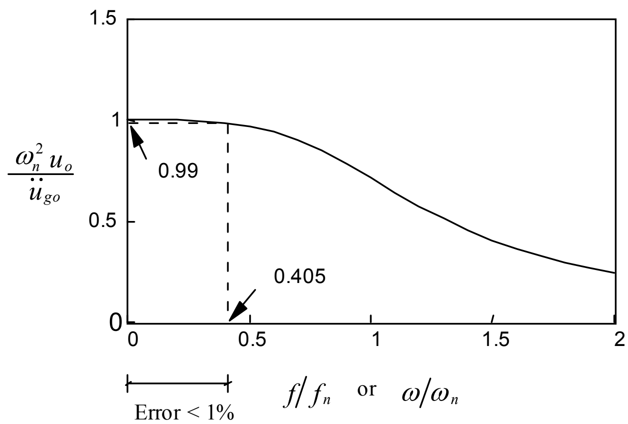
From Eq. (3.7.1),

$$\omega_n^2\, u(t) = -R_d\, \ddot{u}_g(t - \phi/\omega) = -R_d\, \ddot{u}_{go} \sin(2\pi f t - \phi)$$

and therefore,

$$\frac{\omega_n^2 u_o}{\ddot{u}_{go}} = R_d = \frac{1}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(a)}$$

Equation (a) is plotted for $\zeta = 0.7$, the accelerometer damping ratio:



We want to bound R_d as follows

$$0.99 \le R_d \le 1.01 \qquad \text{(b)}$$

The relevant condition is $0.99 \le R_d$ because R_d is always smaller than 1.01 in this case. Therefore, impose

$$\frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} = 0.99 \qquad \text{(c)}$$

Defining $\beta \equiv \omega/\omega_n$, Eq. (c) can be rewritten as

$$(1-\beta^2)^2 + (2\zeta\beta)^2 = \left(\frac{1}{0.99}\right)^2 \Rightarrow$$

$$\beta^4 - 0.04\beta^2 + 1 = 1.0203 \Rightarrow$$

$$\beta^4 - 0.04\beta^2 - 0.0203 = 0 \Rightarrow$$

$$\beta^2 = -0.1239,\quad 0.1639$$

Take only the positive value:

$$\beta^2 = 0.1639 \Rightarrow \beta = 0.405$$

which gives the desired frequency range:

$$f \le 0.405\, f_n = 0.405\,(50) \Rightarrow$$

$$f \le 20.25 \text{ Hz}$$

Problem 3.20

From Eq. (3.7.3),

$$\frac{u_o}{u_{go}} = R_a \qquad \text{(a)}$$

For maximum accuracy, $u_o/u_{go} = 1$; this condition after using Eqs. (3.2.20) and (3.2.11) for R_a gives

$$\frac{(\omega/\omega_n)^2}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} = 1 \qquad \text{(b)}$$

Squaring, simplifying and defining $\beta = \omega/\omega_n$:

$$(1 - \beta^2)^2 + (2\zeta\beta)^2 = \beta^4 \Rightarrow$$

$$1 - 2\beta^2 + 4\zeta^2\beta^2 = 0 \Rightarrow$$

$$\zeta^2 = \frac{1}{2} - \frac{1}{4\beta^2}$$

For $\omega/\omega_n >> 1 \;\; (\beta >> 1)$,

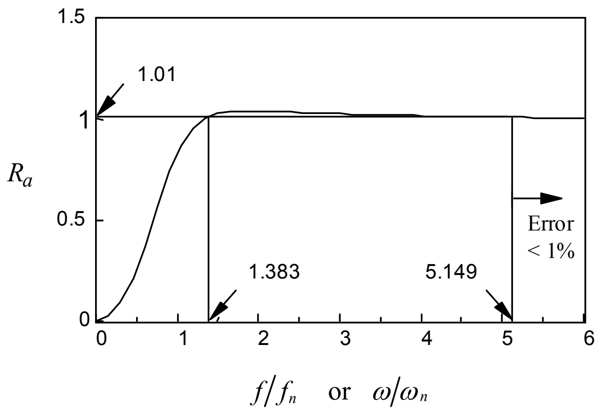
$$\zeta^2 = \frac{1}{2} \Rightarrow \zeta = 0.707$$

Problem 3.21

From Eq. (3.7.3),

$$\frac{u_o}{u_{go}} = R_a = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(a)}$$

Equation (a) is plotted for $\zeta = 0.6$, the instrument damping ratio:



We want to bound R_a as follows (see figure)

$$R_a \leq 1.01 \qquad \text{(b)}$$

Therefore imposing $R_a = 1.01$ which, after defining $\beta \equiv \omega/\omega_n$, gives

$$\frac{\beta^2}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} = 1.01$$

Squaring this equation gives

$$(1 - \beta^2)^2 + (2\zeta\beta)^2 = \left(\frac{\beta^2}{1.01}\right)^2 \Rightarrow$$

$$1 - 2\beta^2 + \beta^4 + 1.44\beta^2 = 0.9803\beta^4 \Rightarrow$$

$$0.0197\beta^4 - 0.56\beta^2 + 1 = 0 \Rightarrow$$

$$\beta^2 = 1.914, \quad 26.51 \Rightarrow \beta = 1.383, \quad 5.149$$

Choose $\beta = 5.149$ (see figure) which gives the desired frequency range:

$$f \geq 5.149 f_n = 5.149\,(0.5) \Rightarrow$$

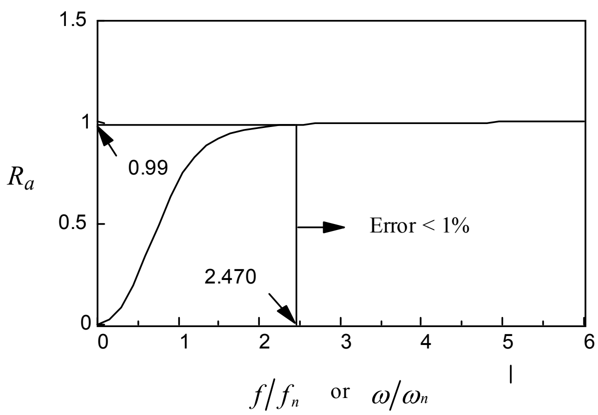
$$f \geq 2.575 \text{ Hz}$$

Problem 3.22

From Eq. (3.7.3),

$$\frac{u_o}{u_{go}} = R_a = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(a)}$$

Equation (a) is plotted for $\zeta = 0.7$, the instrument damping ratio:



Since R_a is always smaller than 1.01, we want to bound R_a as follows

$$R_a \geq 0.99 \qquad \text{(b)}$$

Therefore imposing $R_a = 0.99$ which, after defining $\beta \equiv \omega/\omega_n$, gives

$$\frac{\beta^2}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} = 0.99$$

Squaring this equation gives

$$(1 - \beta^2)^2 + (2\zeta\beta)^2 = \left(\frac{\beta^2}{0.99}\right)^2 \Rightarrow$$

$$1 - 2\beta^2 + \beta^4 + 1.96\beta^2 = 1.0203\beta^4 \Rightarrow$$

$$0.0203\beta^4 + 0.04\beta^2 - 1 = 0 \Rightarrow$$

$$\beta^2 = -8.073, \quad 6.102$$

Take only the positive value:

$$\beta^2 = 6.102 \Rightarrow \beta = 2.470$$

which gives the desired frequency range:

$$f \geq 2.470 f_n = 2.470\,(0.5) \Rightarrow$$

$$f \geq 1.235 \text{ Hz}$$

Problem 3.23

From Eq. (3.8.1),

$$E_D = 2\pi\zeta(\omega/\omega_n)\, ku_o^2 \qquad \text{(a)}$$

where

$$u_o = \frac{p_o}{k} R_d \qquad \text{(b)}$$

In Eq. (a) substituting Eq. (b) and Eq. (3.2.11) for R_d gives

$$E_D = \frac{\pi p_o^2}{k} \frac{2\zeta\omega/\omega_n}{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}$$

Problem 3.24

From Eq. (3.8.9) the loss factor is

$$\xi = \frac{1}{2\pi}\frac{E_D}{E_{S_o}} \qquad \text{(a)}$$

where $E_D = \pi c \omega u_o^2$ and $E_{S_o} = k u_o^2/2$. Substituting these in Eq. (a) gives

$$\xi = \frac{c\,\omega}{k} \qquad \text{(b)}$$

Problem 3.25

Based on equivalent viscous damping, the displacement amplitude is given by

$$\frac{u_o}{(u_{st})_o} = \frac{\left\{1 - \left[(4/\pi)(F/p_o)\right]^2\right\}^{1/2}}{1 - (\omega/\omega_n)^2} \qquad \text{(a)}$$

From the given data

$$\frac{F}{p_o} = \frac{50}{100} = \frac{1}{2}$$

$$\frac{\omega}{\omega_n} = \frac{T_n}{T} = \frac{0.25}{0.3} = 0.833$$

Substituting these data in Eq. (a) gives

$$\frac{u_o}{(u_{st})_o} = \frac{\left\{1 - \left[(4/\pi)(1/2)\right]^2\right\}^{1/2}}{1 - (0.833)^2} = 2.52$$

Now,

$$(u_{st})_o = \frac{p_o}{k}$$

where

$$p_o = 100 \text{ kips}$$

$$k = \omega_n^2 m = \left(\frac{2\pi}{T_n}\right)^2 \frac{w}{\mathrm{g}} = (8\pi)^2 \frac{500}{386}$$
$$= 818 \text{kips/in.}$$

Thus

$$(u_{st})_o = \frac{100}{818} = 0.1222 \text{ in.}$$

and

$$u_o = 2.52\,(0.1222) = 0.308 \text{ in.}$$

Problem 3.26

(a) $p(t)$ is an even function:

$$p(t) = p_o\left(1 - \frac{2}{T_0}t\right) \qquad 0 \le t \le T_0/2 \qquad (a)$$

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} p(t)dt = \frac{2}{T_0}\int_0^{T_0/2} p_o\left(1 - \frac{2}{T_0}t\right)dt$$

$$= \frac{p_o}{2} \qquad (b)$$

$$a_j = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} p(t)\cos(j\omega_0 t)dt$$

$$= \frac{4p_0}{T_0}\int_0^{T_0/2}\left(1 - \frac{2}{T_0}t\right)\cos(j\omega_0 t)dt$$

$$= \frac{4p_0}{T_0}\left\{\frac{1}{j\omega_0}\left[\sin(j\omega_0 t)\right]_0^{T_0/2} - \frac{2}{T_0}\int_0^{T_0/2} t\cos(j\omega_0 t)dt\right\}$$

$$= -\frac{8p_0}{T_0^2}\int_0^{T_0/2} t\cos\left(\frac{2\pi jt}{T_0}\right)dt$$

$$= -\frac{4p_0}{\pi jT_0}\left\{\left[t\sin\left(\frac{2\pi j}{T_0}t\right)\right]_0^{T_0/2} + \frac{T_0}{2\pi j}\left[\cos\left(\frac{2\pi j}{T_0}t\right)\right]_0^{T_0/2}\right\}$$

$$= -\frac{2p_0}{\pi^2 j^2}\left[\cos\left(\frac{2\pi j}{T_0}t\right)\right]_0^{T_0/2}$$

$$= -\frac{2p_0}{\pi^2 j^2}\left[\cos(\pi j) - 1\right]$$

$$\therefore a_j = \begin{cases} \dfrac{4p_o}{\pi^2 j^2} & j = 1,3,5,\cdots \\ 0 & j = 2,4,6,\cdots \end{cases} \qquad (c)$$

$$b_n = 0 \text{ because } p(t) \text{ is an even function} \qquad (d)$$

Thus the Fourier series representation of $p(t)$ is

$$p(t) = \frac{p_o}{2} + \frac{4p_o}{\pi^2}\sum_{j=1,3,5,\cdots}^{\infty}\frac{1}{j^2}\cos(j\omega_0 t) \qquad (e)$$

(b) The steady-state response of an undamped system is obtained by substituting Eqs. (b), (c) and (d) in Eq. (3.13.6) to obtain

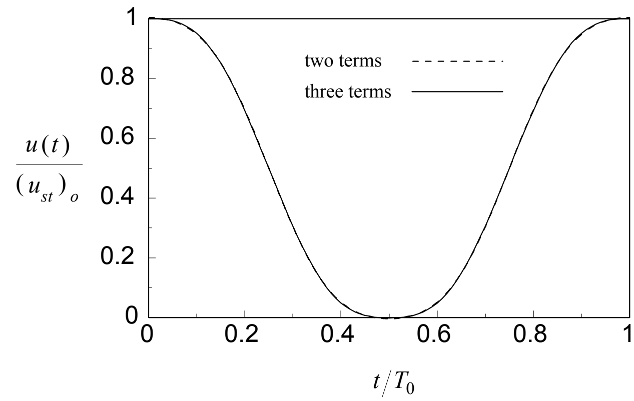
$$\frac{u(t)}{(u_{st})_o} = \frac{1}{2} + \frac{4}{\pi^2}\sum_{j=1,3,5,\cdots}^{\infty}\frac{1}{j^2(1 - \beta_j^2)}\cos(j\omega_0 t)$$

(f)

where $(u_{st})_o = p_o/k$ and $\beta_j = j\omega_0/\omega_n$. Equation (f) is indeterminate when $\beta_j = 1$; corresponding values of T_0 are T_n, $3T_n$, $5T_n$, etc.

(c) For $T_0/T_n = 2$, $\beta_j = j\omega_0/\omega_n = jT_n/T_0 = j/2$ and Eq. (f) becomes

$$\frac{u(t)}{(u_{st})_o} = \frac{1}{2} + \frac{16}{\pi^2}\sum_{j=1,3,5,\cdots}^{\infty}\frac{1}{j^2(4 - j^2)}\cos\left(\frac{2\pi jt}{T_0}\right)$$



Because of the j^4 in the denominator of the series, two terms are enough to obtain reasonable convergence of the series solution.