

## Chapter 2

**2.1** An outcome is the chosen pair of chips. The sample space in this problem consists of 15 pairs: AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, EF (or 30 pairs if the order of chips in each pair matters, i.e., AB and BA are different pairs).

All the outcomes are equally likely because two chips are chosen at random.

One outcome is ‘favorable’, when both chips in a pair are defective (two such pairs if the order matters).

Thus,

$$P(\text{both chips are defective}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \boxed{1/15}$$

**2.2** Denote the events:

$$\begin{aligned} M &= \{ \text{problems with a motherboard} \} \\ H &= \{ \text{problems with a hard drive} \} \end{aligned}$$

We have:

$$P\{M\} = 0.4, \quad P\{H\} = 0.3, \quad \text{and} \quad P\{M \cap H\} = 0.15.$$

Hence,

$$P\{M \cup H\} = P\{M\} + P\{H\} - P\{M \cap H\} = 0.4 + 0.3 - 0.15 = 0.55,$$

and

$$P\{\text{fully functioning MB and HD}\} = 1 - P\{M \cup H\} = \boxed{0.45}$$

**2.3** Denote the events,

$$\begin{aligned} I &= \{ \text{the virus enters through the internet} \} \\ E &= \{ \text{the virus enters through the e-mail} \} \end{aligned}$$

Then

$$\begin{aligned} P\{\bar{E} \cap \bar{I}\} &= 1 - P\{E \cup I\} = 1 - (P\{E\} + P\{I\} - P\{E \cap I\}) \\ &= 1 - (.3 + .4 - .15) = \boxed{0.45} \end{aligned}$$

It may help to draw a Venn diagram.

**2.4** Denote the events,

$$C = \{ \text{knows C/C++} \}, \quad F = \{ \text{knows Fortran} \}.$$

Then

- (a)  $P\{\bar{F}\} = 1 - P\{F\} = 1 - 0.6 = \boxed{0.4}$
- (b)  $P\{\bar{F} \cap \bar{C}\} = 1 - P\{F \cup C\} = 1 - (P\{F\} + P\{C\} - P\{F \cap C\})$   
 $= 1 - (0.7 + 0.6 - 0.5) = 1 - 0.8 = \boxed{0.2}$
- (c)  $P\{C \setminus F\} = P\{C\} - P\{F \cap C\} = 0.7 - 0.5 = \boxed{0.2}$

$$(d) \mathbf{P}\{F \setminus C\} = \mathbf{P}\{F\} - \mathbf{P}\{F \cap C\} = 0.6 - 0.5 = \boxed{0.1}$$

$$(e) \mathbf{P}\{C \mid F\} = \frac{\mathbf{P}\{C \cap F\}}{\mathbf{P}\{F\}} = \frac{0.5}{0.6} = \boxed{0.8333}$$

$$(f) \mathbf{P}\{F \mid C\} = \frac{\mathbf{P}\{C \cap F\}}{\mathbf{P}\{C\}} = \frac{0.5}{0.7} = \boxed{0.7143}$$

**2.5** Denote the events:

$$\begin{aligned} D_1 &= \{\text{first test discovers the error}\} \\ D_2 &= \{\text{second test discovers the error}\} \\ D_3 &= \{\text{third test discovers the error}\} \end{aligned}$$

Then

$$\begin{aligned} \mathbf{P}\{\text{at least one discovers}\} &= \mathbf{P}\{D_1 \cup D_2 \cup D_3\} \\ &= 1 - \mathbf{P}\{\bar{D}_1 \cap \bar{D}_2 \cap \bar{D}_3\} \\ &= 1 - (1 - 0.2)(1 - 0.3)(1 - 0.5) = 1 - 0.28 = \boxed{0.72} \end{aligned}$$

We used the complement rule and independence.

**2.6** Let  $A = \{\text{arrive on time}\}$ ,  $W = \{\text{good weather}\}$ . We have

$$\mathbf{P}\{A \mid W\} = 0.8, \mathbf{P}\{A \mid \bar{W}\} = 0.3, \mathbf{P}\{W\} = 0.6$$

By the Law of Total Probability,

$$\begin{aligned} \mathbf{P}\{A\} &= \mathbf{P}\{A \mid W\} \mathbf{P}\{W\} + \mathbf{P}\{A \mid \bar{W}\} \mathbf{P}\{\bar{W}\} \\ &= (0.8)(0.6) + (0.3)(0.4) = \boxed{0.60} \end{aligned}$$

**2.7** Organize the data. Let  $D = \{\text{detected}\}$ ,  $I = \{\text{via internet}\}$ ,  $E = \{\text{via e-mail}\} = \bar{I}$ . Notice that the question about detection already assumes that the spyware *has entered* the system. This is the sample space, and this is why  $\mathbf{P}\{I\} + \mathbf{P}\{E\} = 1$ . We have

$$\mathbf{P}\{I\} = 0.7, \mathbf{P}\{E\} = 0.3, \mathbf{P}\{D \mid I\} = 0.6, \mathbf{P}\{D \mid E\} = 0.8.$$

By the Law of Total Probability,

$$\mathbf{P}\{D\} = (0.6)(0.7) + (0.8)(0.3) = \boxed{0.66}$$

**2.8** Let  $A_1 = \{\text{1st device fails}\}$ ,  $A_2 = \{\text{2nd device fails}\}$ ,  $A_3 = \{\text{3rd device fails}\}$ .

$$\begin{aligned} \mathbf{P}\{\text{on time}\} &= \mathbf{P}\{\text{all function}\} \\ &= \mathbf{P}\{\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3\} \\ &= \mathbf{P}\{\bar{A}_1\} \mathbf{P}\{\bar{A}_2\} \mathbf{P}\{\bar{A}_3\} && \text{(independence)} \\ &= (1 - 0.01)(1 - 0.02)(1 - 0.02) && \text{(complement rule)} \\ &= \boxed{0.9508} \end{aligned}$$

$$2.9 \quad P\{\text{at least one fails}\} = 1 - P\{\text{all work}\} = 1 - (.96)(.95)(.90) = \boxed{0.1792}.$$

$$2.10 \quad P\{A \cup B \cup C\} = 1 - P\{\bar{A} \cap \bar{B} \cap \bar{C}\} = 1 - P\{\bar{A}\}P\{\bar{B}\}P\{\bar{C}\} \\ = 1 - (1 - 0.4)(1 - 0.5)(1 - 0.2) = \boxed{0.76}$$

$$2.11 \quad (a) \quad P\{\text{at least one test finds the error}\} \\ = 1 - P\{\text{all tests fail to find the error}\} \\ = 1 - (1 - 0.1)(1 - 0.2)(1 - 0.3)(1 - 0.4)(1 - 0.5) \\ = 1 - (0.9)(0.8)(0.7)(0.6)(0.5) = \boxed{0.8488}$$

(b) The difference between events in (a) and (b) is the probability that *exactly one* test finds an error. This probability equals

$$P\{\text{exactly one test finds the error}\} \\ = P\{\text{test 1 find the error, the others don't find}\} \\ + P\{\text{test 2 find the error, the others don't find}\} + \dots \\ = (0.1)(1 - 0.2)(1 - 0.3)(1 - 0.4)(1 - 0.5) \\ + (1 - 0.1)(0.2)(1 - 0.3)(1 - 0.4)(1 - 0.5) + \dots \\ = (0.1)(0.8)(0.7)(0.6)(0.5) + (0.9)(0.2)(0.7)(0.6)(0.5) \\ + (0.9)(0.8)(0.3)(0.6)(0.5) + (0.9)(0.8)(0.7)(0.4)(0.5) \\ + (0.9)(0.8)(0.7)(0.6)(0.5) = 0.3714.$$

Then

$$P\{\text{at least two tests find the error}\} \\ = P\{\text{at least one test finds the error}\} \\ - P\{\text{exactly one test finds the error}\} \\ = 0.8488 - 0.3714 = \boxed{0.4774}$$

$$(c) \quad P\{\text{all tests find the error}\} = (0.1)(0.2)(0.3)(0.4)(0.5) = \boxed{0.0012}$$

2.12 Let  $A_j = \{\text{dog } j \text{ detects the explosives}\}$ .

$$P\{\text{at least one dog detects}\} = 1 - P\{\text{all four dogs don't detect}\} \\ = 1 - P\{\bar{A}_1\}P\{\bar{A}_2\}P\{\bar{A}_3\}P\{\bar{A}_4\} \\ = 1 - (1 - 0.6)^4 = \boxed{0.9744}$$

2.13 Let  $A_j$  be the event  $\{\text{Team } j \text{ detects a problem}\}$ . Then

$$P\{\text{at least one team detects}\} = 1 - P\{\text{no team detects}\} \\ = 1 - P\{\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3\} = 1 - P\{\bar{A}_1\}P\{\bar{A}_2\}P\{\bar{A}_3\} \\ = 1 - (1 - 0.8)(1 - 0.8)(1 - 0.8) = \boxed{0.992}.$$

2.14 (a) The total number of possible passwords is

$$P(26, 6) = (26)(25)(24)(23)(22)(21) = 165,765,600$$

because there are 26 letters in the alphabet, they should be all different in the

password, and the order of characters is important. The password is guessed (favorable outcome) if it is among the 1,000,000 attempted passwords. Then

$$\begin{aligned} P\{\text{guess the password}\} &= \frac{\text{number of favorable passwords}}{\text{total number of passwords}} \\ &= \frac{1,000,000}{165,765,600} = \boxed{0.0060} \end{aligned}$$

- (b) Now we can use 52 characters, and the order is still important. Then the total number of passwords is

$$P(52, 6) = (52)(51)(50)(49)(48)(47) = 14,658,134,400,$$

and

$$P\{\text{guess the password}\} = \frac{1,000,000}{14,658,134,400} = \boxed{0.000068}$$

- (c) Letters can be repeated in passwords, therefore, the total number of passwords is

$$P_r(52, 6) = 52^6,$$

and

$$P\{\text{guess the password}\} = \frac{10^6}{52^6} = \boxed{0.000051}$$

- (d) Adding the digits brings the number of possible characters to 62. Then the total number of passwords is

$$P_r(62, 6) = 62^6,$$

and

$$P\{\text{guess the password}\} = \frac{10^6}{62^6} = \boxed{0.000018}$$

The more characters we use the lower is the probability for a spyware to break into the system.

- 2.15** Let  $A = \{\text{Error in the 1st block}\}$  and  $B = \{\text{Error in the 2nd block}\}$ . Then  $P\{A\} = 0.2$ ,  $P\{B\} = 0.3$ , and  $P\{A \cap B\} = 0.06$  by independence;  
 $P\{\text{error in program}\} = P\{A \cup B\} = 0.2 + 0.3 - 0.06 = 0.44$ .

Then, by the definition of conditional probability,

$$P\{A \cap B \mid A \cup B\} = \frac{P\{A \cap B\}}{P\{A \cup B\}} = \frac{0.06}{0.44} = \boxed{0.1364}$$

Or, by the Bayes Rule,

$$\begin{aligned} P\{A \cap B \mid A \cup B\} &= \frac{P\{A \cup B \mid A \cap B\} P\{A \cap B\}}{P\{A \cup B\}} \\ &= \frac{(1)(0.06)}{0.44} = \boxed{0.1364} \end{aligned}$$

- 2.16** Organize the data. Let  $D = \{\text{defective part}\}$ . We are given:

$$\begin{array}{l|l} P\{S1\} = 0.5 & P\{D|S1\} = 0.05 \\ P\{S2\} = 0.2 & P\{D|S2\} = 0.03 \\ P\{S3\} = 0.3 & P\{D|S3\} = 0.06 \end{array}$$

We need to find  $P\{S1|D\}$ .

(a) By the Law of Total Probability:

$$\begin{aligned} P\{D\} &= P\{D|S1\}P\{S1\} + P\{D|S2\}P\{S2\} + P\{D|S3\}P\{S3\} \\ &= (0.5)(0.05) + (0.2)(0.03) + (0.3)(0.06) = \boxed{0.049} \end{aligned}$$

(b) Bayes Rule:

$$P\{S1|D\} = \frac{P\{D|S1\}P\{S1\}}{P\{D\}} = \frac{(0.5)(0.05)}{0.049} = \boxed{25/49 \text{ or } 0.5102}$$

**2.17** Let  $D = \{\text{defective part}\}$ . We are given:

$$\begin{array}{l|l} P\{X\} = 0.24 & P\{D|X\} = 0.05 \\ P\{Y\} = 0.36 & P\{D|Y\} = 0.10 \\ P\{Z\} = 0.40 & P\{D|Z\} = 0.06 \end{array}$$

Combine the Bayes Rule and the Law of Total Probability.

$$\begin{aligned} P\{Z|D\} &= \frac{P\{D|Z\}P\{Z\}}{P\{D|X\}P\{X\} + P\{D|Y\}P\{Y\} + P\{D|Z\}P\{Z\}} \\ &= \frac{(0.06)(0.40)}{(0.05)(0.24) + (0.10)(0.36) + (0.06)(0.40)} \\ &= \boxed{1/3 \text{ or } 0.3333} \end{aligned}$$

**2.18** Let  $C = \{\text{correct}\}$ ,  $G = \{\text{guessing}\}$ . It is given that:

$$P\{\bar{G}\} = 0.75, \quad P\{C|\bar{G}\} = 0.9, \quad P\{C|G\} = 1/4 = 0.25.$$

Also,  $P\{G\} = 1 - 0.75 = 0.25$ .

Then, by the Bayes Rule,

$$\begin{aligned} P\{G|C\} &= \frac{P\{C|G\}P\{G\}}{P\{C|G\}P\{G\} + P\{C|\bar{G}\}P\{\bar{G}\}} \\ &= \frac{(0.25)(0.25)}{(0.25)(0.25) + (0.9)(0.75)} = \boxed{0.0847} \end{aligned}$$

**2.19** Let  $D = \{\text{defective part}\}$  and  $I = \{\text{inspected electronically}\}$ . By the Bayes Rule,

$$\begin{aligned} P\{I|D\} &= \frac{P\{D|I\}P\{I\}}{P\{D|I\}P\{I\} + P\{D|\bar{I}\}P\{\bar{I}\}} \\ &= \frac{(1 - 0.95)(0.20)}{(1 - 0.95)(0.20) + (1 - 0.7)(1 - 0.20)} = \boxed{0.0400} \end{aligned}$$

**2.20** Let  $S = \{\text{steroid user}\}$  and  $N = \{\text{test is negative}\}$ .

It is given that  $P\{S\} = 0.05$ ,  $P\{\bar{N}|S\} = 0.9$ , and  $P\{\bar{N}|\bar{S}\} = 0.02$ .

By the complement rule,  $P\{\bar{S}\} = 0.95$ ,  $P\{N|S\} = 0.1$ , and  $P\{N|\bar{S}\} = 0.98$ .

By the Bayes Rule,

$$\begin{aligned} P\{S|N\} &= \frac{P\{N|S\}P\{S\}}{P\{N|S\}P\{S\} + P\{N|\bar{S}\}P\{\bar{S}\}} \\ &= \frac{(0.1)(0.05)}{(0.1)(0.05) + (0.98)(0.95)} = \boxed{5/936 \text{ or } 0.00534} \end{aligned}$$

**2.21** At least one of the first three components works with probability

$$1 - P\{\text{all three fail}\} = 1 - (0.3)^3 = 0.973.$$

At least one of the last two components works with probability

$$1 - P\{\text{both fail}\} = 1 - (0.3)^2 = 0.91.$$

Hence, the system operates with probability  $(0.973)(0.91) = \boxed{0.8854}$

**2.22** (a) The scheme of cities A, B, and C and all five highways is similar to Exercise 2.21. Similarly to this exercise, there exists an open route from city A to city C with probability

$$\{1 - (0.2)^3\} \{1 - (0.2)^2\} = \boxed{0.9523}$$

(b- $\alpha$ ) If the new highway is built between cities A and B, it will be the 4-th highway connecting A and B. Then the probability of an open route from city A to city C becomes

$$\{1 - (0.2)^4\} \{1 - (0.2)^2\} = \boxed{0.9585}$$

(b- $\beta$ ) If the new highway is built between B and C, it will be the 3rd highway connecting these cities. Then the probability of an open route from city A to city C is

$$\{1 - (0.2)^3\} \{1 - (0.2)^3\} = \boxed{0.9841}$$

(b- $\gamma$ ) Finally, if the new highway is built between A and C, then  $P\{\text{at least one open route from A to C}\}$

$$\begin{aligned} &= \left\{ \begin{array}{l} \text{a new direct route} \\ \text{from A to C is open} \end{array} \cup \left\{ \begin{array}{l} \text{a route from A to B to C} \\ \text{is open, see question (a)} \end{array} \right\} \right\} \\ &= 1 - (1 - 0.2)(1 - 0.9523) = \boxed{0.9618} \end{aligned}$$

**2.23** (a)  $(0.9)(0.8) = \boxed{0.72}$

(b)  $1 - \{1 - (0.9)(0.8)\} \{1 - (0.7)(0.6)\} = \boxed{0.8376}$

(c)  $1 - (1 - 0.9)(1 - 0.8)(1 - 0.7) = \boxed{0.994}$

(d)  $\{1 - (1 - 0.9)(1 - 0.7)\} \{1 - (1 - 0.8)(1 - 0.6)\} = \boxed{0.8924}$

(e)  $\{1 - (1 - 0.9)(1 - 0.6)\} \{1 - (1 - 0.8)(1 - 0.7)(1 - 0.5)\} = \boxed{0.9312}$

**2.24** A customer is unaware of defects, so he buys 6 random laptops. The outcomes are equally likely, so each probability can be computed as

$$\frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

$$(a) \mathbf{P}\{\text{exactly } 2\} = \frac{\binom{5}{2} \binom{5}{4}}{\binom{10}{6}} = \frac{\frac{5 \cdot 4}{2} \cdot 5}{\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}} = \boxed{\frac{5}{21} \text{ or } 0.238}$$

(b) This is a conditional probability because  $\{X \geq 2\}$  is given. We need

$$\mathbf{P}\{X = 2 \mid X \geq 2\} = \frac{\mathbf{P}\{X = 2 \cap X \geq 2\}}{\mathbf{P}\{X \geq 2\}} = \frac{\mathbf{P}\{X = 2\}}{\mathbf{P}\{X \geq 2\}}$$

where  $\mathbf{P}\{X = 2\} = 5/21$  is already computed in (a), and

$$\mathbf{P}\{x \geq 2\} = 1 - P(X = 1) = 1 - \frac{\binom{5}{1} \binom{5}{5}}{\binom{10}{6}} = 1 - \frac{5 \cdot 1}{\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}} = \frac{41}{42}$$

Notice that  $\mathbf{P}\{X = 0\} = 0$  because there are only 5 good computers, so among the purchased 6 computers there has to be at least 1 defective. So,

$$\mathbf{P}\{X = 2 \mid X \geq 2\} = \frac{\mathbf{P}\{X = 2\}}{\mathbf{P}\{X \geq 2\}} = \frac{5/21}{41/42} = \boxed{10/41 \text{ or } 0.244}.$$

**2.25** Our sample space consists of birthdays of all  $N = 30$  students. The total number of outcomes in it is

$$\mathcal{N}_T = P_r(365, N) = 365^N.$$

It is easier to count the outcomes where all students are born on *different* days. The number of such outcomes is

$$\mathcal{N}_F = P(365, N) = \frac{365!}{(365 - N)!} = (365)(364) \dots (365 - N).$$

Then

$$\begin{aligned} P(N) &= \mathbf{P}\{\text{at least two share birthdays}\} \\ &= 1 - \mathbf{P}\{\text{all born on different days}\} \\ &= 1 - \frac{\mathcal{N}_F}{\mathcal{N}_T} = 1 - \left(\frac{365}{365}\right) \left(\frac{364}{365}\right) \dots \left(\frac{365 - N}{365}\right). \end{aligned}$$

For  $N = 30$ , we get

$$P(30) = 1 - 0.2937 = \boxed{0.7063}$$

(b) Evaluating  $P(N)$  for different  $N$ , we see that  $P(22) = 0.4757$  and  $P(23) = 0.5073$ . Hence, we need at least  $\boxed{23}$  students in order to find birthday matches with a probability above 0.5.

**2.26** The sample space consists of all (unordered) sets of three computers selected from six computers in the lab. Favorable outcomes are sets of three computers with non-defective hard drives. We have

$$\mathcal{N}_T = C(6, 3) = \frac{(6)(5)(4)}{(3)(2)(1)} = 20 \quad \text{and} \quad \mathcal{N}_F = C(4, 3) = 4;$$

therefore,

$$\mathbf{P}\{\text{no hard drive problems}\} = \frac{\mathcal{N}_F}{\mathcal{N}_T} = \frac{4}{20} = \boxed{0.2}$$

- 2.27** The sample space consists of all unordered sets of five computers selected from 18 computers in the store. Favorable outcomes are sets of five non-defective computers (that come from a subset of  $18 - 6 = 12$ ). Then

$$\mathcal{N}_T = C(18, 5) = \frac{(18)(17)(16)(15)(14)}{(5)(4)(3)(2)(1)} \quad \text{and} \quad \mathcal{N}_F = C(12, 5) = \frac{(12)(11)(10)(9)(8)}{(5)(4)(3)(2)(1)};$$

therefore,

$$P \left\{ \begin{array}{l} \text{five computers} \\ \text{without defects} \end{array} \right\} = \frac{\mathcal{N}_F}{\mathcal{N}_T} = \frac{(12)(11)(10)(9)(8)}{(18)(17)(16)(15)(14)} = \boxed{\frac{11}{119} \text{ or } 0.0924}$$

- 2.28** The sample space consists of sequences of 6 answers where each answer is one of 4 possible answers, say, A, B, C, or D. Then a sequence of 6 answers is a 6-letter word written with letters A, B, C, and D with replacement. The student guesses, therefore, all outcomes are equally likely.

The total number of outcomes is

$$\mathcal{N}_T = P_r(4, 6) = 4^6 = 4096.$$

Favorable outcomes occur when the student guesses at least 3 answers correctly. This includes 3, 4, 5, and 6 correct answers. The correctly answered questions are chosen at random from 6 questions. Then, a correct answer is given to each of the chosen questions. Also, an incorrect answer to each remaining question is chosen out of 3 possible incorrect answers. Altogether, the number of favorable outcomes is

$$\begin{aligned} \mathcal{N}_F &= C(6, 3)(3^3) + C(6, 4)(3^2) + C(6, 5)(3^1) + C(6, 6)(3^0) \\ &= \frac{(6)(5)(4)}{(3)(2)(1)}(27) + \frac{(6)(5)}{(2)(1)}(9) + (6)(3) + 1 = 694. \end{aligned}$$

$$P \{\text{he will pass}\} = \frac{\mathcal{N}_F}{\mathcal{N}_T} = \frac{694}{4096} = \boxed{0.1694}$$

One can also use the complement rule for a little shorter solution.

- 2.29** Outcomes are sets of four databases selected from nine. Favorable outcomes are such sets where at least 2 databases have a keyword, out of 5 such databases (and the remaining ones don't have a keyword, so they come from the remaining 4 databases). Then

$$\mathcal{N}_T = C(9, 4) = \frac{(9)(8)(7)(6)}{(4)(3)(2)(1)} = 126,$$

$$\begin{aligned} \mathcal{N}_F &= C(5, 2)C(4, 2) + C(5, 3)C(4, 1) + C(5, 4)C(4, 0) \\ &= (10)(6) + (10)(4) + (5)(1) = 105, \end{aligned}$$

and

$$P \{\text{at least two have the keyword}\} = \frac{\mathcal{N}_F}{\mathcal{N}_T} = \frac{105}{126} = \boxed{\frac{5}{6} \text{ or } 0.8333}$$

- 2.30 (a) All outcomes are listed in the table below. According to the problem, they are equally likely.

Outcome	The older child	The younger child	Who is met
1	girl	girl	the older girl
2	girl	girl	the younger girl
3	girl	boy	the girl
4	girl	boy	the boy
5	boy	girl	the girl
6	boy	girl	the boy
7	boy	boy	the older boy
8	boy	boy	the younger boy

- (b)  $P\{BB\} = P\{\text{outcomes 7, 8}\} = 1/4$ ,  
 $P\{BG\} = P\{\text{outcomes 5, 6}\} = 1/4$ ,  
 $P\{GB\} = P\{\text{outcomes 3, 4}\} = 1/4$ .
- (c) Meeting Jimmy automatically eliminates outcomes 1, 2, 3, and 5. The remaining outcomes are

Outcome	The older child	The younger child	Who is met
4	girl	boy	the boy
6	boy	girl	the boy
7	boy	boy	the older boy
8	boy	boy	the younger boy

Two remaining outcomes form the event  $BB$  whereas  $BG$  and  $GB$  have only one outcome each. Therefore, given that you met a boy,

$$\begin{aligned} P\{BB \mid \text{met Jimmy}\} &= P\{\text{outcomes 7, 8} \mid \text{met Jimmy}\} = 1/2, \\ P\{BG \mid \text{met Jimmy}\} &= P\{\text{outcome 6} \mid \text{met Jimmy}\} = 1/4, \\ P\{GB \mid \text{met Jimmy}\} &= P\{\text{outcome 4} \mid \text{met Jimmy}\} = 1/4. \end{aligned}$$

- (d)  $P\{\text{Jimmy has a sister} \mid \text{met Jimmy}\}$   
 $= P\{\text{outcomes 4, 6} \mid \text{met Jimmy}\}$   
 $= 1/2$ .

- 2.31 According to (2.2),

$$\overline{A \cap B \cap C \cap \dots} = \overline{A} \cup \overline{B} \cup \overline{C} \cup \dots$$

Then, events  $A, B, C, \dots$  are disjoint (i.e.,  $A \cap B \cap C \cap \dots = \emptyset$ ) if and only if

$$\overline{A} \cup \overline{B} \cup \overline{C} \cup \dots = \overline{A \cap B \cap C \cap \dots} = \overline{\emptyset} = \Omega.$$

We see that the union of  $\overline{A}, \overline{B}, \overline{C}, \dots$  equals the entire sample space in this case. By Definition 2.8  $\overline{A}, \overline{B}, \overline{C}, \dots$  are exhaustive.

- 2.32 *Intuitive solutions:*

- (a) Independent events  $A$  and  $B$  occur independently of each other. Hence, they *don't occur* independently of each other. Every time when  $A$  (or  $B$ ) does not occur, its complement occurs. Hence, the complements of  $A$  and  $B$  are also independent of each other.

- (b) Being disjoint is a very strong dependence because disjoint events completely eliminate each other. The only way for such events to be independent is when one of these events is *always* eliminated. Such an event must have probability 0.
- (c) Being exhaustive is also a strong type of dependence because one event absolutely has to cover all the parts of  $\Omega$  that are not covered by the other event. The only way for such events to be independent is when one of the events covers all the parts of  $\Omega$  regardless of the other event. Such event should be the entire sample space,  $\Omega$ .

*Mathematical solutions:*

- (a) Using (2.2),

$$\begin{aligned}
 \mathbf{P}\{\bar{A} \cap \bar{B}\} &= \mathbf{P}\{\overline{A \cup B}\} = 1 - \mathbf{P}\{A \cup B\} \\
 &= 1 - (\mathbf{P}\{A\} + \mathbf{P}\{B\} - \mathbf{P}\{A \cap B\}) \\
 &= 1 - \mathbf{P}\{A\} - \mathbf{P}\{B\} + \mathbf{P}\{A\} \mathbf{P}\{B\} \\
 &\quad \text{(because } A \text{ and } B \text{ are independent)} \\
 &= (1 - \mathbf{P}\{A\})(1 - \mathbf{P}\{B\}) \\
 &= \mathbf{P}\{\bar{A}\} \mathbf{P}\{\bar{B}\}.
 \end{aligned}$$

Hence,  $\bar{A}$  and  $\bar{B}$  are independent.

- (b) If  $A$  and  $B$  are independent and disjoint, then

$$0 = \mathbf{P}\{A \cap B\} = \mathbf{P}\{A\} \mathbf{P}\{B\},$$

which can only happen when  $\mathbf{P}\{A\} = 0$  or  $\mathbf{P}\{B\} = 0$ .

- (c) If  $A$  and  $B$  are independent and exhaustive, then

$$\begin{aligned}
 1 &= \mathbf{P}\{A \cup B\} = \mathbf{P}\{A\} + \mathbf{P}\{B\} - \mathbf{P}\{A \cap B\} \\
 &= \mathbf{P}\{A\} + \mathbf{P}\{B\} - \mathbf{P}\{A\} \mathbf{P}\{B\}.
 \end{aligned}$$

Then

$$0 = 1 - (\mathbf{P}\{A\} + \mathbf{P}\{B\} - \mathbf{P}\{A\} \mathbf{P}\{B\}) = (1 - \mathbf{P}\{A\})(1 - \mathbf{P}\{B\}),$$

which can only happen when  $\mathbf{P}\{A\} = 1$  or  $\mathbf{P}\{B\} = 1$ .

**2.33** Generalizing (2.4), we prove that for any events  $E_1, \dots, E_n$ ,

$$\begin{aligned}
 &\mathbf{P}\{E_1 \cup \dots \cup E_n\} \\
 &= \sum_{i \leq n} \mathbf{P}\{E_i\} - \sum_{1 \leq i < j \leq n} \mathbf{P}\{E_i \cap E_j\} + \sum_{1 \leq i < j < k \leq n} \mathbf{P}\{E_i \cap E_j \cap E_k\} - \dots \\
 &\quad - (-1)^n \mathbf{P}\{E_1 \cap \dots \cap E_n\}.
 \end{aligned}$$

This can be proved by induction.

For  $n = 2$  events, this formula is given by (2.4).

Suppose the formula is true for  $n$  events. Let  $A$  denote their overall union,  $A = E_1 \cup \dots \cup E_n$ . Then for any event  $E_{n+1}$ ,

$$\mathbf{P}\{E_1 \cup \dots \cup E_{n+1}\} = \mathbf{P}\{A \cup E_{n+1}\}$$

$$\begin{aligned}
&= \mathbf{P}\{A\} + \mathbf{P}\{E_{n+1}\} - \mathbf{P}\{A \cap E_{n+1}\} \\
&= \sum_{i \leq n+1} \mathbf{P}\{E_i\} - \sum_{1 \leq i < j \leq n} \mathbf{P}\{E_i \cap E_j\} + \sum_{1 \leq i < j < k \leq n} \mathbf{P}\{E_i \cap E_j \cap E_k\} \\
&\quad - \dots - (-1)^n \mathbf{P}\{E_1 \cap \dots \cap E_n\} - \mathbf{P}\{A \cap E_{n+1}\}.
\end{aligned}$$

Also, since the formula is assumed true for  $n$  events,

$$\begin{aligned}
\mathbf{P}\{A \cap E_{n+1}\} &= \mathbf{P}\{(E_1 \cap E_{n+1}) \cup \dots \cup (E_n \cap E_{n+1})\} \\
&= \sum_{i \leq n} \mathbf{P}\{E_i \cap E_{n+1}\} - \sum_{1 \leq i < j \leq n} \mathbf{P}\{E_i \cap E_j \cap E_{n+1}\} + \dots \\
&\quad - (-1)^n \mathbf{P}\{E_1 \cap \dots \cap E_{n+1}\}.
\end{aligned}$$

Altogether,

$$\begin{aligned}
&\mathbf{P}\{E_1 \cup \dots \cup E_{n+1}\} \\
&= \sum_{i \leq n+1} \mathbf{P}\{E_i\} - \sum_{1 \leq i < j \leq n+1} \mathbf{P}\{E_i \cap E_j\} + \sum_{1 \leq i < j < k \leq n+1} \mathbf{P}\{E_i \cap E_j \cap E_k\} \\
&\quad - \dots - (-1)^{n+1} \mathbf{P}\{E_1 \cap \dots \cap E_{n+1}\}.
\end{aligned}$$

This proves the formula for  $(n+1)$  events. By induction, the formula is proved for any  $n \geq 2$ .

**2.34** Let  $A_i = \bar{E}_i$  for  $i = 1, \dots, n$ . According to (2.2),

$$\bar{A}_1 \cap \dots \cap \bar{A}_n = \overline{A_1 \cup \dots \cup A_n}$$

Therefore,

$$\overline{E_1 \cap \dots \cap E_n} = \overline{\bar{A}_1 \cap \dots \cap \bar{A}_n} = A_1 \cup \dots \cup A_n = \bar{E}_1 \cup \dots \cup \bar{E}_n$$

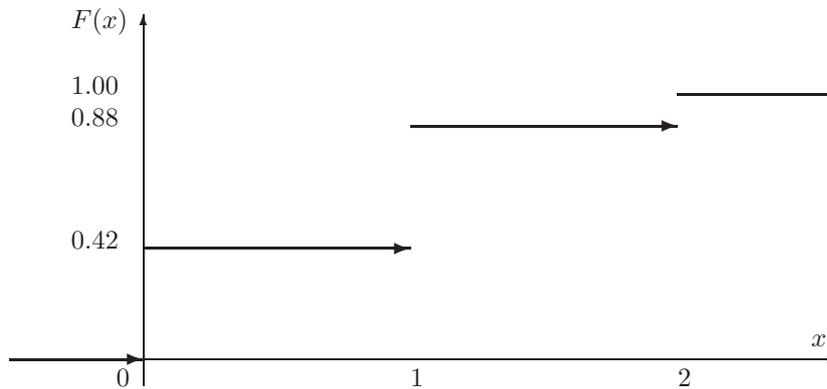
**2.35** Events  $A \setminus B$  and  $B$  are mutually exclusive, and their union is  $A$ . Therefore,  $\mathbf{P}\{A \setminus B\} + \mathbf{P}\{B\} = \mathbf{P}\{A\}$ , and  $\mathbf{P}\{A \setminus B\} = \mathbf{P}\{A\} - \mathbf{P}\{B\}$ .

**2.36** Consider the following events,

$$A_1 = E_1, A_2 = E_2 \setminus E_1, A_3 = E_3 \setminus (E_1 \cup E_2), A_4 = E_4 \setminus (E_1 \cup E_2 \cup E_3), \dots$$

They are mutually exclusive,  $A_i \subset E_i$  for all  $i$ , and  $A_1 \cup A_2 \cup \dots = E_1 \cup E_2 \cup \dots$ . Then,

$$\mathbf{P}\{E_1 \cup E_2 \cup \dots\} = \mathbf{P}\{A_1 \cup A_2 \cup \dots\} = \sum_i \mathbf{P}\{A_i\} \leq \sum_i \mathbf{P}\{E_i\}.$$

FIGURE 1: The cdf of  $X$  for Exercise 3.1

## Chapter 3

**3.1** Possible values of  $X$  are: 0, 1, and 2.

(a) The pmf is:

$$\begin{aligned}
 P(0) &= \mathbf{P}\{\text{both files are not corrupted}\} \\
 &= (1 - 0.4)(1 - 0.3) = 0.42, \\
 P(1) &= \mathbf{P}\left\{\begin{array}{l} \text{1st is corrupted,} \\ \text{2nd is not} \end{array}\right\} + \mathbf{P}\left\{\begin{array}{l} \text{2nd is corrupted,} \\ \text{1st is not} \end{array}\right\} \\
 &= (0.4)(1 - 0.3) + (0.3)(1 - 0.4) = 0.46, \\
 P(2) &= \mathbf{P}\{\text{both are corrupted}\} = (0.4)(0.3) = 0.12.
 \end{aligned}$$

(check:  $P(0) + P(1) + P(2) = 1$ .)

(b) The cdf is given in Figure 1.

**3.2** Let  $X$  be the number of network blackouts, and  $Y$  be the loss. Then  $Y = 500X$ . Compute

$$\begin{aligned}
 \mathbf{E}(X) &= \sum_x xP(x) = (0)(0.7) + (1)(0.2) + (2)(0.1) = 0.4; \\
 \text{Var}(X) &= \sum_x (x - 0.4)^2 P(x) \\
 &= (0 - 0.4)^2(0.7) + (1 - 0.4)^2(0.2) + (2 - 0.4)^2(0.1) = 0.44.
 \end{aligned}$$

Hence,

$$\mathbf{E}(Y) = 500 \mathbf{E}(X) = (500)(0.4) = \boxed{200 \text{ dollars}}$$

and

$$\text{Var}(Y) = 500^2 \text{Var}(X) = (250,000)(0.44) = \boxed{110,000 \text{ squared dollars}}$$