Answer 1.2
(a) The minimum is at $r = a = 2^{1/6}\sigma$ where the potential takes the value $V(a) = -\epsilon$. **(b)** Expanding to second order in $r - a$ we get $V(r) = \frac{1}{2}k(r-a)^2$ with $k = 72\epsilon/a^2$. **(c)** The frequency of harmonic radial oscillations is $\omega = \sqrt{k/m}$ where m is the mass of the atom. For Argon one finds $k = 1.45\ \mathrm{J/m^2}$ and $\omega = 4.7 \times 10^{12}\ \mathrm{s^{-1}}$.

Problem 1.3 Consider a collection of N identical molecules (a "material particle") taken from a large volume of gas. Let the instantaneous molecular velocities be $\boldsymbol{v}_n$ for $n = 1, 2 \cdots, N$. Collisions with other molecules in the gas at large will randomly change the velocity of each of the selected molecules, but if there is no overall drift in the gas, the velocity of individual molecules should average out to zero, $\langle \boldsymbol{v}_n \rangle = \boldsymbol{0}$, the velocities of different molecules should be uncorrelated, $\langle \boldsymbol{v}_n \boldsymbol{v}_m \rangle = \boldsymbol{0}$ for $n \neq m$, and the average of the square of the velocity should be the same for all molecules, $\left\langle \boldsymbol{v}_n^2 \right\rangle = v_0^2$.

(a) Show that the root-mean-square average of the center-of-mass velocity of the collection equals $v_0/\sqrt{N}$.

Answer 1.3 **(a)** The CMS-velocity is $\boldsymbol{v} = N^{-1} \sum_n \boldsymbol{v}_n$, and the average of its square becomes

$$\left\langle \boldsymbol{v}^2 \right\rangle = \frac{1}{N^2} \sum_{n,m} \langle \boldsymbol{v}_n \cdot \boldsymbol{v}_m \rangle = \frac{1}{N^2} \sum_n \left\langle \boldsymbol{v}_n^2 \right\rangle = \frac{1}{N^2} \sum_n v_0^2 = \frac{1}{N} v_0^2.$$

Problem 1.4 Any distance function must satisfy the axioms

$$\begin{aligned} &d(\boldsymbol{a}, \boldsymbol{a}) = 0, && \\ &d(\boldsymbol{a}, \boldsymbol{b}) = d(\boldsymbol{b}, \boldsymbol{a}), && \text{(symmetry)} \\ &d(\boldsymbol{a}, \boldsymbol{b}) \leq d(\boldsymbol{a}, \boldsymbol{c}) + d(\boldsymbol{c}, \boldsymbol{b}). && \text{(triangle inequality)} \end{aligned}$$

Show that the Cartesian distance function (1.14) satisfies these axioms.

Answer 1.4 The two first inequalities are trivial. In the third we may without loss of generality place the coordinate system such that $\boldsymbol{c} = \boldsymbol{0}$, $\boldsymbol{a} = (a_1, 0, 0)$, and $\boldsymbol{b} = (b_1, b_2, 0)$. The third inequality then becomes $\sqrt{(a_1 - b_1)^2 + b_2^2} \leq \sqrt{a_1^2} + \sqrt{b_1^2 + b_2^2}$. Squaring it and canceling common terms on both sides, it becomes $-2a_1 b_1 \leq 2\,|a_1|\,\sqrt{b_1^2 + b_2^2}$. It is obviously fulfilled for $a_1 b_1 \geq 0$, while for $a_1 b_1 < 0$ the inequality may be squared once more to yield $4a_1^2 b_1^2 \leq 4a_1^2(b_1^2 + b_2^2)$ which is trivially fulfilled.

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2 Pressure

Problem 2.1 The normal human systolic blood pressure is usually quoted as 120 mm mercury (above atmospheric pressure). A clinical *sphygomanometer* used to measure blood pressure is constructed from a U-tube half filled with mercury. During measurement, the air pressure in one arm of the manometer is supplied by an inflated blood-constricting cuff around the upper arm or the wrist, whereas the other arm of the manometer is exposed to atmospheric pressure. **(a)** How long should the arms of the manometer be when it must accommodate a measurement range of $\pm 100\%$ around normal?

Answer 2.1 **(a)** The range of the manometer is from 0 to 240 mmHg. The distance between the pressurized and open mercury surfaces must therefore at least be 240 mm = 24 cm. This is indeed the typical size of clinical mercury manometers.

Problem 2.2 Consider a canal with a dock gate that is 12 m wide and has water depth 9 m on one side and 6 m on the other side. Calculate

(a) The pressure in the water on both sides of the gate at a height z over the bottom.
(b) The total force on the gate.
(c) The total moment of force around the bottom of the gate.
(d) The height over the bottom at which the total force acts.

Answer 2.2 Put $h_1 = 9$ m, $h_2 = 6$ m and $a = 12$ m. Atmospheric pressure is p_0.
(a) On one side $p_1 = p_0 + \rho_0 g_0 (h_1 - z)$, on the other $p_2 = p_0 + \rho_0 g_0 (h_2 - z)$.
(b) $\mathcal{F}_1 = \int_0^{h_1} (p_1 - p_0) a \, dz = \frac{1}{2} h_1^2 a \rho_0 g_0$. $\mathcal{F} = \mathcal{F}_1 - \mathcal{F}_2 \approx 2.7 \times 10^6$ N.
(c) $\mathcal{M}_1 = \int_0^{h_1} z(p_1 - p_0) a \, dz = \frac{1}{6} h_1^3 a \rho_0 g_0$. $\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2 \approx 10^7$ Nm.
(d) $z = \mathcal{M}/\mathcal{F} = 3.8$ m.

Problem 2.3 An underwater lamp is covered by a hemispherical glass with a radius of $a = 15$ cm and is placed with its center at a depth of $h = 3$ m on the side of the pool. **(a)** Calculate the total horizontal force from the water on the lamp when there is air at normal pressure inside.

Answer 2.3 The horizontal pressure force on the hemisphere must be equal to the pressure force on the vertical plane through the center of the sphere. The linear rise of pressure with depth makes the hydrostatic pressure act with its average value $\Delta p = \rho_0 g_0 h$ at the center. So the horizontal force becomes $\Delta p \pi a^2 \approx 2100$ N, corresponding to the weight of 210 kg.

If you do not like this argument, it is also possible with some effort to integrate the pressure force directly in spherical coordinates with the x-axis orthogonal to the wall,

$$\begin{aligned}
\boldsymbol{\mathcal{F}} &= -\int_{\text{half-sphere}} (p - p_0)\, d\boldsymbol{S} = -\int_{\text{half-sphere}} (p - p_0) \hat{\boldsymbol{e}}_r \, dS \\
&= -a^2 \int_0^{\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi \, \sin\theta \, (p - p_0)\, \hat{\boldsymbol{e}}_r .
\end{aligned}$$

Using $p = p_0 + \rho_0 g_0 (h - a\cos\theta)$, the x-component of the force becomes

$$\begin{aligned}
\mathcal{F}_x &= -a^2 \int_0^{\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi \, \sin\theta \, \rho_0 g_0 (h - a\cos\theta) \sin\theta \cos\phi \\
&= -2a^2 \rho_0 g_0 \int_0^{\pi} d\theta \sin^2\theta (h - a\cos\theta) = -\rho_0 g_0 h \pi a^2 .
\end{aligned}$$

Problem 2.4 Using a manometer, the pressure in an open container filled with liquid is found to be 1.6 atm at a height of 6 m over the bottom, and 2.8 atm at a height of 3 m. **(a)** Determine the density of the liquid and **(b)** the height of the liquid surface.

Answer 2.4 Put $h_1 = 6$ m, $p_1 = 1.6$ atm and $h_2 = 3$ m, $p_2 = 2.8$ atm. From $p_1 - p_2 = -\rho_0 g_0 (h_1 - h_2)$ we get $\rho_0 \approx 4100$ kg m^{-3}, and from $p_1 - p_0 = -(h_1 - h_0)\rho_0 g_0$ we get $h_0 = 7.5$ m.

Problem 2.5 An open jar contains two non-mixable liquids with densities $\rho_1 > \rho_2$. The heavy layer has thickness h_1 and the light layer on top of it has thickness h_2. **(a)** An open glass tube is now lowered vertically into the liquids toward the bottom of the jar. Describe how high the liquids rise in the tube (disregarding capillary effects). **(b)** The open tube is already placed in the container with its opening close to the bottom when the heavy fluid is poured in, followed by the light. How high will the heavy fluid rise in the tube?

Answer 2.5 **(a)** The surface inside the tube will be at the same level $h_1 + h_2$ as in the jar. **(b)** The heavy liquid in the tube must initially rise to the same level h_1 as in the jar. When the light liquid is poured in, the surface of the heavy liquid in the tube must rise further to balance the weight of the light and rise to a height $h_1 + h_2\rho_2/\rho_1 < h_1 + h_2$.

Problem 2.6 Show that a mixture of ideal gases (see page 4) also obeys the equation of state (2.27).

Answer 2.6 Repeating the argument leading to (2.7) the pressure becomes the sum over molecular species, $p = \frac{1}{3}\sum_i \rho_i \left\langle \boldsymbol{v}^2 \right\rangle_i$ where $\rho_i = N_i m_i / V$ is the density of each species. Consequently, by the equipartition theorem $pV = \frac{1}{3}\sum_i N_i m_i \left\langle \boldsymbol{v}^2 \right\rangle_i = \sum_i N_i k_B T = N k_B T = n R_{\text{mol}} T$ where $n = N/N_A = M/M_{\text{mol}} = \rho V/M_{\text{mol}}$ is the total number of moles, ρ is the average density and M_{mol} the average molar mass.

Problem 2.7 The equation of state due to van der Waals is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT,$$

where a and b are constants. It describes gases and their condensation into liquids. **(a)** Calculate the isothermal bulk modulus. **(b)** Under which conditions can it become negative, and what does that mean?

Answer 2.7 Solving for the pressure, we find

$$P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}.$$

(a) Differentiating, we get

$$K_T = -V\left(\frac{\partial p}{\partial V}\right)_T = \frac{nRTV}{(V - nb)^2} - \frac{2an^2}{V^2}.$$

(b) It can become negative for sufficiently low temperature, satisfying

$$RT < \frac{2an(V - nb)^2}{V^3},$$

which means that the gas must have condensed.

Problem 2.8 Calculate the pressure and density in the flat-Earth sea, assuming constant bulk modulus. **(a)** Show that both quantities are singular at a certain depth and calculate this depth.

Answer 2.8 **(a)** When K is constant it follows from (2.42) that $p = K \log \rho + \text{const}$. Inserting this into the local hydrostatic equations (2.23), it may be solved with the result,

$$\rho = \frac{\rho_0}{1 + \dfrac{z}{h_1}}, \qquad p = p_0 - K \log\left(1 + \frac{z}{h_1}\right),$$

where

$$h_1 = \frac{K}{\rho_0 g_0} = \frac{K}{p_0} h_0.$$

The pressure and density become singular for $z = -h_1$, which for water is $h_1 \approx 235$ km.

$*$ **Problem 2.9** Calculate the isentropic scale height for the Mars atmosphere.

Answer 2.9 Typical data for Mars are $g_0 = 3.73$ m s^{-1}, $T_0 = 220$ K, $M_{\rm mol} = 44$ g mol^{-1}, and $\gamma = 4/3$ (carbon dioxide). Thus $h_2 = \frac{\gamma}{\gamma-1}\frac{RT_0}{g_0} \approx 45$ km.

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3 Buoyancy and stability

Problem 3.1 A stone weighs $\mathcal{F}_1 = 1{,}000$ N in vacuum and $\mathcal{F}_0 = 600$ N when submerged in water of density ρ_0. **(a)** Calculate the volume V and **(b)** average density ρ_1 of the stone.

Answer 3.1 **(a)** The weight of the displaced water is $\mathcal{F}_1 - \mathcal{F}_0 = \rho_0 V g_0$, so that $V = (\mathcal{F}_1 - \mathcal{F}_0)/\rho_0 g_0 = 0.04$ m^3. **(b)** The weight of the stone is $\mathcal{F}_0 = \rho_1 V g_0$ so that $\rho_1 = \mathcal{F}_0/V g_0 = g_0\rho_0\mathcal{F}_1/(\mathcal{F}_1 - \mathcal{F}_0) = 2500$ kg m^{-3}.

Problem 3.2 A hydrometer is an instrument used to measure the density of a liquid. A certain hydrometer with mass $M = 4$ g consists of a roughly spherical bulb and a long thin cylindrical stem of radius $a = 2$ mm. The sphere is weighed down so that the apparatus will float stably with the stem pointing vertically upward and crossing the fluid surface at some point. **(a)** How much deeper will it float in alcohol with mass density $\rho_1 = 0.78$ g cm^{-3} than in oil with mass density $\rho_2 = 0.82$ g cm^{-3}? You may disregard the tiny density of air.

Answer 3.2 **(a)** The mass of the hydrometer displaces two different volumes $M = \rho_1 V_1 = \rho_2 V_2$. The difference in volumes is only due to the change in the extra piece of the stem below the waterline in the lighter alcohol, or $\pi a^2 d = V_2 - V_1 = M(1/\rho_2 - 1/\rho_1)$. The result is $d = 20$ mm.

Problem 3.3 A cylindrical wooden stick with density $\rho_1 = 0.65$ g cm^{-3} floats in water (density $\rho_0 = 1$ g cm^{-3}). The stick is loaded down with a lead weight with density $\rho_2 = 11$ g cm^{-3} at one end such that it floats in a vertical orientation with a fraction $f = 1/10$ of its length out of the water. **(a)** What is the ratio M_1/M_2 between the masses of the wooden stick and the lead weight? **(b)** As a function of the density of the wood, how large a fraction of the stick can be out of the water in hydrostatic equilibrium (disregarding questions of stability)?

Answer 3.3 **(a)** Displacement $M_1 + M_2 = \rho_0((1-f)V_1 + V_2)$ with $M_1 = \rho_1 V_1$ and $M_2 = \rho_2 V_2$. Solving for the volume ratio

$$\frac{V_1}{V_2} = \frac{\rho_2 - \rho_0}{(1-f)\rho_0 - \rho_1},$$

the mass ratio becomes $M_1/M_2 = 2.36$.
(b) The denominator must be positive: $f < 1 - \rho_1/\rho_0 = 0.35$.

Problem 3.4 Prove without assuming constant gravity that the hydrostatic moment of buoyancy equals (minus) the moment of gravity of the displaced fluid (corollary to Archimedes' principle).