CHAPTER 2 The Nature of Sets

Chapter Overview
This chapter introduces some basic building blocks for mathematics, namely the notion of sets and set theory This chapter will be particular important in Chapter 3 (logic) and Chapter 13 (probability), but the terminology of sets is found in every chapter of this book. Remember, the focus of this book is everyday usage of mathematics, so it is important to relate the everyday (nonmathematical) usage of the words *and, or*, and *not* to their mathematical counterparts, *intersection, union*, and *complement.* These same words will be discussed in a logical context in the next chapter.

Chapter Challenge: The sum of the numbers in the first numbers in each row is entered into the third entry in that same row, so $2 + 7 = 9$ and $5 + 4 = 9$. Thus, the correct value for the question mark the third row is $7 + 11 = 18$. You might ask why this problem appears at the beginning of this chapter. OK, Chapter 1 is about problem solving, so the Chapter Challenge for Chapter 1 made sense, but why this problem, here in Chapter 2, which is a chapter on sets? What does this problem have to do with sets? Well, we answer this by saying that this book is about problem solving, and these Chapter Challenge problems will appear at the beginning of every chapter because it is worthwhile to see challenge problems *outside* a chapter context. Different instructors will use these challenge problems in different ways; some will use them to begin the chapter, some will use them as extra credit at the end of the chapter, while others may assign them as homework problems. This is an example of a magic square, which was discussed in Section 1.2.

2.1 Sets, Subsets, and Venn Diagrams, page 49

New Terms Introduced in this Section

Belongs to	Cardinal number	Cardinality	Circular definition
Complement	Contained in	Counting number	Description method
Disjoint	Element	Empty set	Equal sets
Equivalent sets	Improper subset	Integer	Member
Natural number	Proper subset	Rational number	Roster method
Set	Set-builder notation	Set theory	Subset
Universal set	Venn diagram	Well-defined set	Whole number

Teaching Suggestions

The ideas of this section are sometimes difficult for the students, particularly ideas involving the empty set. I have found that students can understand the material more easily if I use { } instead of $\varnothing$. The idea that the empty set is a subset of every set as well as the difference between $\varnothing$ and $\{\varnothing\}$ need some very careful development. A device that I have used to make it easy for the students to see the difference between an element of a set and a subset of a set is to bring a bunch of little boxes of cutout cardboard pieces representing elements; the boxes represent sets and the pieces of cardboard represent elements. We then discuss the ideas of this section using these boxes as models. This might seem like a juvenile demonstration, but the students need something concrete to look at, and I've found that they understand the material much better when I use this model.

Level 1, page 55

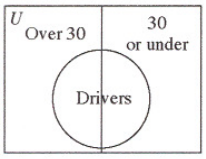
1. Answers vary. Not every word can be defined, so in order to build a mathematical system, there must be a starting place, with some words given as undefined. **2.** Answers vary. *Equal* sets refer to sets that are identical, whereas *equivalent* sets refer to sets with the same cardinality. Equal sets are always equivalent, but equivalent sets do not need to be equal sets. **3.** Answers vary. **a.** The universal set is the set that includes all the elements under consideration for a particular discussion. **b.** The empty set is the set with cardinality 0; that is the set containing no elements. **4.** Answers vary. **a.** Set of people over 10 feet tall; set of negative integers between 0 and 1; set of living cats with seven heads. **b.** Set of people over 20 ft tall. **5. a.** well defined **b.** not well defined **6. a.** well defined **b.** not well defined **7. a.** not well defined **b.** not well defined **8. a.** well defined **b.** not well defined **9. a.** $\{m, a, t, h, e, i, c, s\}$ **b.** Until 20{Barack Obama} **10. a.** $\{1, 3, 5, 7, 9\}$ **b.** $\{3, 6, 9, \cdots\}$ **11. a**. $\{7, 8, 9, 10, \cdots\}$ **b.** $\{1, 2, 3, 4, 5\}$ **12. a**. $\{7, 8, 9, 10, \cdots\}$ **b.** $\{\cdots, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ **13. a.** $\{p, i, e\}$ **b.** $\{151, 152, 153, \cdots\}$ **14. a.** $\{1, 11, 111, 1111, \cdots\}$ **b.** $\{6, 8, 10, 12, 14\}$ *Answers to Problems 15–20 may vary.* **15.** counting numbers less than 10 **16.** nonnegative powers of 11 **17.** multiples of 10 between 0 and 101 **18.** 5×10^n for n a counting number **19.** odd numbers between 100 and 170 **20.** distinct letters in the word *Mississippi* **21.** The set of all numbers, x, such that x is an odd counting number; $\{1, 3, 5, 7, \cdots\}$ **22.** The set of all natural numbers, x such that x is between 1 and 10; $\{2, 3, 4, 5, 6, 7, 8, 9\}$ **23.** The set of natural numbers, x, such that x is not equal to 8; $\{1, 2, 3, 4, 5, 6, 7, 9, 10, \cdots\}$ **24.** The set of whole numbers, x, such that x is less than or equal to eight; $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ **25.** The set of all whole numbers, x, such that x is less than eight; $\{0, 1, 2, 3, 4, 5, 6, 7\}$ **26.** The set of all whole numbers, x, except for the even counting numbers; $\{0, 1, 3, 5, 7, \cdots\}$

	Description	Roster	Set-builder
27.	all even numbers	$\{\cdots, -4, -2, 0, 2, 4, \cdots\}$	$\{x \mid x = 2k,\ k \text{ an integer}\}$
28.	all positive multiples of 5	$\{5, 10, 15, \cdots\}$	$\{x \mid x = 5n, n \text{ a counting number}\}$
29.	all counting numbers between 0 and 10	$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$	$\{n \mid 0 < n < 10, n \text{ a counting number}\}$
30.	positive multiples of 3	$\{3, 6, 9, 12, \cdots\}$	$\{x \mid x = 3n, n \text{ a counting number}\}$
31.	multiples of 5	$\{\cdots, -10, -5, 0, 5, \cdots\}$	$\{x \mid x = 5k,\ k \text{ an integer}\}$
32.	vowels	$\{A, E, I, O, U\}$	$\{x \mid x \text{ is a vowel}\}$
33.	all positive multiples of 4	$\{4, 8, 12, \cdots\}$	$\{x \mid x = 4n,\ n \text{ a counting number}\}$
34.	counting numbers between -5 and 5, inclusive	$\{-5, -4, -3, \cdots, 3, 4, 5\}$	$\{n \mid -5 \le n \le 5, n \text{ a counting number}\}$
35.	counting numbers greater than 100	$\{101, 102, 103, \cdots\}$	$\{n \mid n > 100,\ n \text{ a counting number}\}$

36. a. $\varnothing$ **b.** $\varnothing$, $\{1\}$ **c.** $\varnothing$, $\{1\}$, $\{2\}$, $\{1, 2\}$ **d.** $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$ **e.** $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2, 3, 4\}$ **37. a.** $\varnothing$ **b.** $\varnothing$, $\{6\}$ **c.** $\varnothing$, $\{6\}$, $\{7\}$, $\{6, 7\}$ **d.** $\varnothing$, $\{6\}$, $\{7\}$, $\{8\}$, $\{6, 7\}$, $\{6, 8\}$, $\{7, 8\}$, $\{6, 7, 8\}$ **e.** $\varnothing$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, $\{6, 7\}$, $\{6, 8\}$, $\{6, 9\}$, $\{7, 8\}$, $\{7, 9\}$, $\{8, 9\}$, $\{6, 7, 8\}$, $\{6, 7, 9\}$, $\{6, 8, 9\}$, $\{7, 8, 9\}$, $\{6, 7, 8, 9\}$ **38.** 32 subsets; yes **39.** 32 subsets; yes

Level 2, page 56

40.



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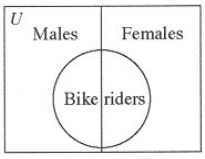

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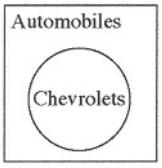

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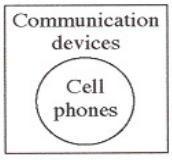


44. a. $|A| = 3$, $|B| = 1$, $|C| = 3$, $|D| = 1$, $|E| = 1$, $|F| = 1$ **b.** $A \leftrightarrow C$; $B \leftrightarrow D \leftrightarrow E \leftrightarrow F$ **c.** $A = C$; $D = E = F$ **45. a.** $|A| = 1$, $|B| = 1$, $|C| = 2$, $|D| = 1$, $|E| = 2$ **b.** $A \leftrightarrow B \leftrightarrow D$; $C \leftrightarrow E$ **c.** $A = B$ **46. a.** true **b.** false **47. a.** true **b.** false **48. a.** true **b.** false **49. a.** false **b.** true **50. a.** true **b.** true **51. a.** true **b.** false **52. a.** false **b.** true **53. a.** true **b.** false **54. a.** true **b.** false

Level 3, page 57

55. Answers vary; the set of real numbers, or the set of people alive in China today.

56. Answers vary; yes. First, we list the halves: $\frac{1}{2}$; then the thirds: $\frac{1}{3}$, $\frac{2}{3}$; followed by the fourths: $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$; but wait, $\frac{2}{4} = \frac{1}{2}$, so it has already been listed. Now, move through the fifths, sixths, seventh, eights, $\cdots$ leaving out all of the ones previously listed:

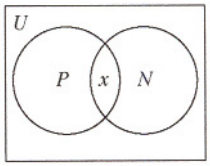
$$\left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \cdots\right\}$$

This problem anticipates what is done in the text in Section 2.4. **57.** There will be five different ways, since each group of four will have one missing person. **58.** List the possibilities; there are 10.

Level 3 Problem Solving, page 57

59. The set of Chevrolets is a subset of the set of automobiles.

60. Let $U = \{\text{all creatures}\}$, $P = \{\text{people}\}$, and $N = \{\text{nice creatures}\}$. The region marked "$x$" means that region is not empty.



2.2 Operations with Sets, page 57

New Terms Introduced in this Section

And Intersection Or Union

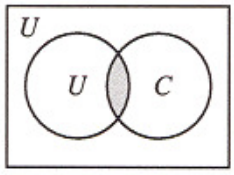
Teaching Suggestions

In this section we introduce the operations of intersection and union, along with their Venn diagrams (see Transparency 8), and we review complementation.

I introduce this day's class using the question is again considered in Problem Set 2.2, Problem 2. The formula for the cardinality of a union should also be discussed in class because this with be used later in the text.

Level 1, page 59

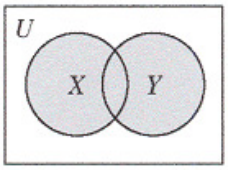
1. *Union*: $A \cup B$ is the set of all elements in either A or B (or possibly in both). *Intersection:* $A \cap B$ is the set of all elements in both A and B. *Complement*: $\overline{A}$ is the set of all elements not in A. **2.** Answers vary; Let U = students accepted at the public university; C = those accepted at the local private college. If James adds the number of elements in U and the number of elements in C, he will have counted the number of elements in the intersection (shaded portion) twice.



For example, suppose 200 students are considered and we find that 100 are accepted at the public university, and 100 are accepted at the local private college. If 30 of these people are accepted at both schools, then we see that a total of 170 have been accepted at one of these

schools, leaving 30 people not accepted at all. **3. a.** or **b.** and **c.** not **4. a.** $|X \cap Y|$ **b.** $|X \cup Y| = |X| + |Y| - |X \cap Y|$ **5.** It is true when $X \cap Y = \varnothing$. **6.** It is always true. **7. a.** $\{2, 6, 8, 10\}$ **b.** $\{6, 8\}$ **c.** $\{1, 3, 4, 5, 7, 9, 10\}$ **8. a.** $\{2, 3, 5, 6, 8, 9\}$ **b.** $\varnothing$ **c.** $\{1, 3, 4, 5, 6, 7, 10\}$ **9. a.** $\{3, 4, 5\}$ **b.** $\{1, 2, 3, 4, 5, 6, 7\}$ **c.** $\{6, 7, 8, 9, 10\}$ **10. a.** $\varnothing$ **b.** $\{1, 2, 5, 7, 9\}$ **c.** $\{3, 4, 6, 8, 10\}$ **11.** $\{1, 3, 5, 7, 9\}$ **12.** $\{2, 4, 6, 8, 10\}$ **13.** $\{2, 3, 4, 6, 8, 9, 10\}$ **14.** $\{6\}$ **15.** $\{3, 5, 6, 9, 10\}$ **16.** $\varnothing$ **17.** $\{1, 3, 5, 7, 9\}$ **18.** $\{2, 4, 6, 8, 10\}$ **19.** $\{x | x$ is a nonzero integer$\}$ **20.** $\varnothing$ **21. a.** $\mathbb{W}$ **b.** $\mathbb{N}$ **22. a.** $\varnothing$ **b.** U **23. a.** $\varnothing$ **b.** X **24. a.** U **b.** $\varnothing$ **25.** $\{1, 2, 3, 4, 5, 6\}$ **26.** $\{1, 2\}$ **27.** $\{1, 2, 3, 4, 5, 7\}$ **28.** $\{3\}$ **29.** $\{5\}$ **30.** $\{1, 2, 3, 5, 6, 7\}$ **31.** $\{5, 6, 7\}$ **32.** $\{3, 4, 7\}$ **33.** $\{1, 2, 4, 6\}$ **34.** $\varnothing$ **35.** $\{1, 2, 3, 4\}$ **36.** $\{1, 2, 3, 4\}$ **37.** $\{1, 2, 3, 4, 6, 7\}$ **38.** $\varnothing$ **39.** A **40.** $\varnothing$ **41.** $A \cap B$ **42.** $A \cup B$ **43.** $\overline{B}$ **44.** $\overline{A}$ **45.** $\overline{A \cap B}$ **46.** $\overline{A \cup B}$

47.



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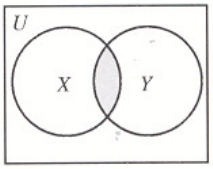

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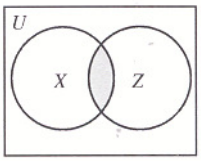

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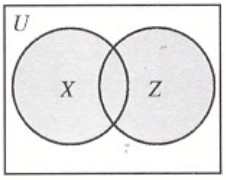

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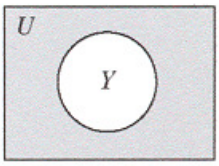

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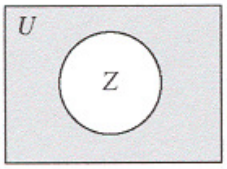


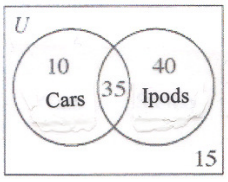
Level 2, page 60

53. We work this problem without Venn diagrams to show an alternate method of solution. Let $B = \{$people in band$\}$ and $S = \{$people in orchestra$\}$.

$$|B \cup S| = |B| + |S| - |B \cap S|$$
$$= 50 + 36 - 14$$
$$= 72$$

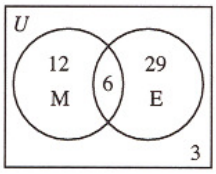
Yes, they can get by with two buses.

54.



a. 10 + 40 = 50 **b.** 15

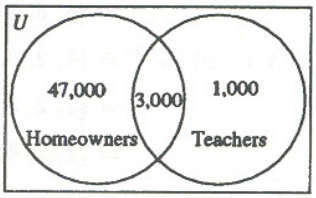
55.



Three people are not taking either.

56. $|F \cup B| = |F| + |B| - |B \cap F|$ There are 34 people playing.
$= 25 + 16 - 7$
$= 34$

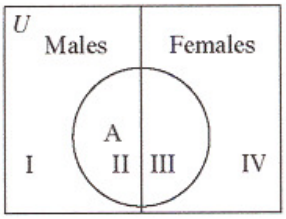
57.



From the Venn diagram, we see 51,000 booklets are needed.

58. Let U = {SRJC students}, F = {females}, and A = {students over 25};

$|U| = 29{,}000$;
$|F| = 0.58(29{,}000) = 16{,}820$;
$|\overline{F}| = 29{,}000 - 16{,}820 = 12{,}180$;
$|A| = 0.62(29{,}000) = 17{,}980$;
$|\overline{F} \cap A| = 0.40(17{,}980) = 7{,}192$
$|F \cap A| = 0.60(17{,}980) = 10{,}788$



Thus, I: $12{,}180 - 7{,}192 = 4{,}988$ II: 7,192 III: 10,788 IV: $16{,}820 - 10{,}788 = 6{,}032$

Level 3 Problem Solving, page 61

59. We use patterns;
1 set is $2 = 2^1$ regions (inside and outside); 2 sets are $4 = 2^2$; 3 sets are $8 = 2^3$
It looks like 4 sets should be $2^4 = 16$ and 5 sets should be $2^5 = 32$. Our hypothesis for n sets is 2^n regions.

60. This is a puzzle problem. We proceed by trial and error. Remember, that the only possible answers for the letters A through I are 1 through 9.
If $A = 1$, then $B = 8$, and if $B = 8$, then $C = -1$; this is not correct.
If $A = 2$, then $B = 7$, and if $B = 7$, then $C = 0$; this is not correct.
If ***A = 3***, then ***B = 6***, and if $B = 6$, then ***C = 1***; this is possible.
If $B = 6$, then ***E = 8***.

If $E = 8$, then **$G = 9$**.
No clues for H, so we look at $F + I = 11$.
If $F = 2$, then $I = 9$; this has been used.
If **$F = 4$**, then **$I = 7$** and **$D = 5$**.
This leaves one letter and one number unused: **$H = 2$.**

Thus, the answer is: $A = 3$, $B = 6$, $C = 1$, $D = 5$, $E = 8$, $F = 4$, $G = 9$, $H = 2$, and $I = 7$.
By the way, there is one other possibility, and that is when $F = 7$. If this value is used, then $D = 2$, $I = 4$, which forces $H = 5$.

2.3 Applications of Sets, page 61

New Terms Introduced in this Section

Associative property for union and intersection De Morgan's laws

Teaching Suggestions

With combined operations with sets, I think it is worthwhile to practice verbalizing statements such as "complement of a union" (Ans: $\overline{X \cup Y}$) or "union of complement" (Ans: $\overline{X} \cup \overline{Y}$). The *process* for proving a set statement is illustrated quite nicely by proving De Morgan's laws. If you intend on covering the logic chapter, there is a nice tie between this section and logical statements. Even though I don't always work textbook examples in class, I find it worthwhile to do Example 3 in class. Many students have trouble with these combined operations, and spending some time with this example will pay dividends later.

Survey problems are the fun part of this section. You can use examples or problems from the text, or you can also conduct some surveys (with three sets) on your own and use those as a springboard for class discussion.

Level 1, page 66

1. *De Morgan's laws* relate the operations of complements, unions, and intersections. There are two statements: $\overline{X \cup Y} = \overline{X} \cap \overline{Y}$ and $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$.

2. Answers vary. Draw a Venn diagram, and then fill in the number of elements in the various regions. Fill in the innermost region first, and then work your way outward (using subtraction) until the number of elements of the eight regions formed by the three sets is known.

3. $$\begin{aligned}(A \cup B) \cap C &= (\{1, 2, 3, 4\} \cup \{1, 2, 5, 6\}) \cap \{3, 5, 7\} \\ &= \{1, 2, 3, 4, 5, 6\} \cap \{3, 5, 7\} \\ &= \{3, 5\}\end{aligned}$$

4. $A \cup (B \cap C) = \{1, 2, 3, 4\} \cup (\{1, 2, 5, 6\} \cap \{3, 5, 7\})$
$= \{1, 2, 3, 4\} \cup \{5\}$
$= \{1, 2, 3, 4, 5\}$

5. $\overline{A \cup B} \cap C = \overline{\{1, 2, 3, 4, 5, 6\}} \cap \{3, 5, 7\}$
$= \{7\} \cap \{3, 5, 7\}$
$= \{7\}$

6. $A \cup \overline{B \cap C} = \{1, 2, 3, 4\} \cup \overline{\{1, 2, 5, 6\} \cap \{3, 5, 7\}}$
$= \{1, 2, 3, 4\} \cup \overline{\{5\}}$
$= \{1, 2, 3, 4\} \cup \{1, 2, 3, 4, 6, 7\}$
$= \{1, 2, 3, 4, 6, 7\}$

7. $\overline{A} \cup (B \cap C) = \overline{\{1, 2, 3, 4\}} \cup \{5\}$
$= \{5, 6, 7\} \cup \{5\}$
$= \{5, 6, 7\}$

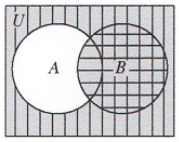
8. $(A \cup B) \cap \overline{C} = \{1, 2, 3, 4, 5, 6\} \cap \overline{\{3, 5, 7\}}$
$= \{1, 2, 3, 4, 5, 6\} \cap \{1, 2, 4, 6\}$
$= \{1, 2, 4, 6\}$

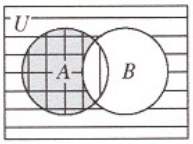
9. $\overline{(A \cup B) \cap C} = \overline{\{3, 5\}}$ *From Problem 3.*
$= \{1, 2, 4, 6, 7\}$

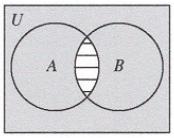
10. $\overline{A} \cup (\overline{B} \cap \overline{C}) = \overline{\{1, 2, 3, 4\}} \cup (\overline{\{1, 2, 5, 6\}} \cap \overline{\{3, 5, 7\}})$
$= \{5, 6, 7\} \cup (\{3, 4, 7\} \cap \{1, 2, 4, 6\})$
$= \{5, 6, 7\} \cup \{4\}$
$= \{4, 5, 6, 7\}$

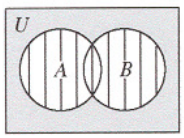
11. $\overline{X} \cup \overline{Y}$ **12.** $\overline{X \cup Y}$ **13.** $\overline{X \cap Y}$ **14.** $\overline{X} \cap \overline{Y}$ **15.** $\overline{X \cup Y}$ **16.** $\overline{X} \cup \overline{Y}$
17. $\overline{X} \cap \overline{Y}$ **18.** $\overline{X \cap Y}$

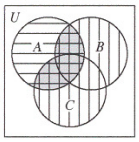
Level 2, page 66

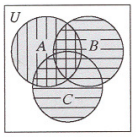
19. 

20. 

21. 

22. 

23. 

24. 

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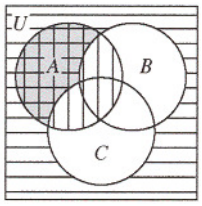

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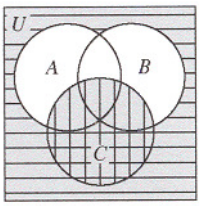

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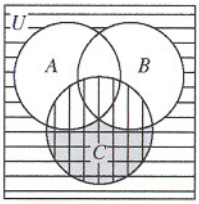

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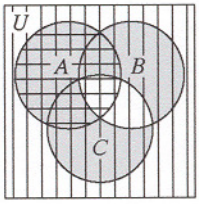

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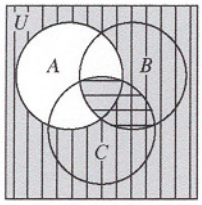

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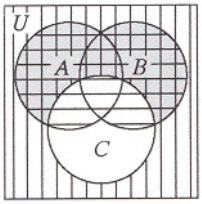

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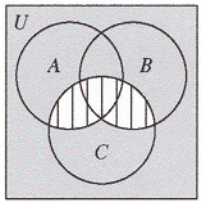

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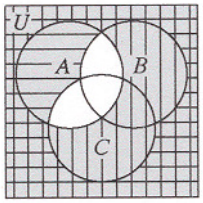
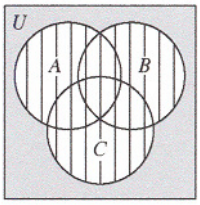
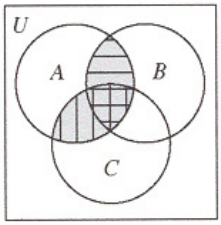


33.



34.



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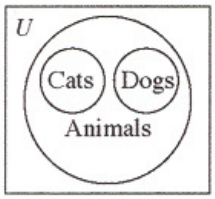

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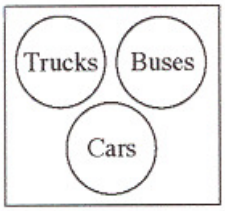

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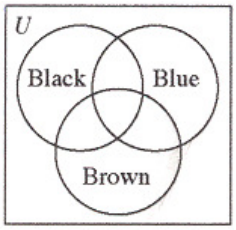

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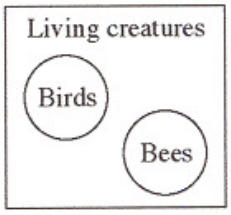
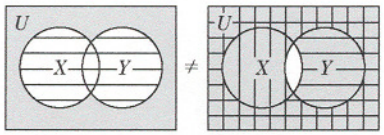


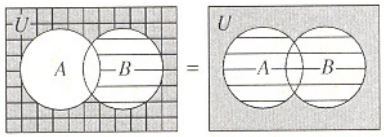
The order of operations in Problems 39–44 may vary.

39. $\overline{A \cup B}$ **40.** $\overline{A} \cap B$ **41.** $A \cap (B \cup C)$ **42.** $B \cap (A \cup C)$

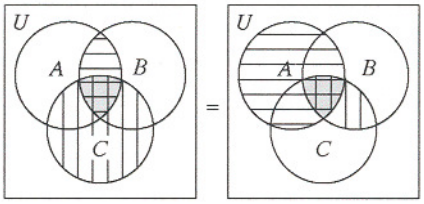
43. $A \cup (B \cap C)$ **44.** $B \cup (A \cap C)$

45. 

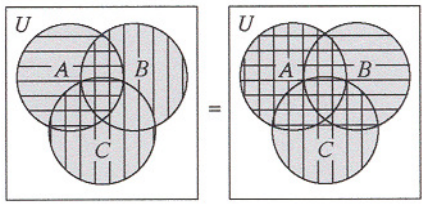
It is false.

46. 

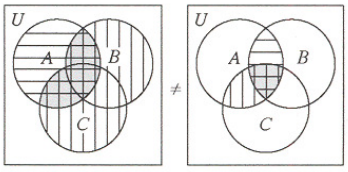
It is true.

47. 

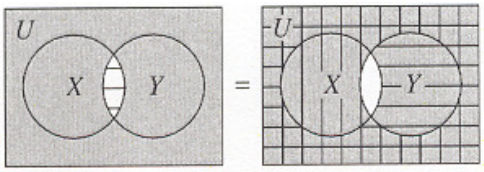
It is true.

48. 

It is true.

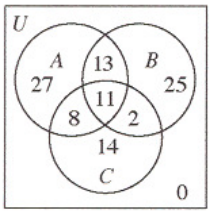
49. 

It is false.

50. 

It is true.

51. Let A = {use shampoo A}; B = {use shampoo B}; C = {use shampoo C}

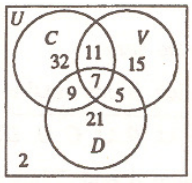
Draw the Venn diagram. 

Begin with region for all three sets, and fill in 11.
Fill in the remaining part of $A \cap B$: $24 - 11 = 13$.
Fill in the remaining part of $A \cap C$: $19 - 11 = 8$.
Fill in the remaining part of $B \cap C$: $13 - 11 = 2$.
Next, fill in the remaining parts for each of the sets: A: $59 - 13 - 11 - 8 = 27$
B: $51 - 13 - 11 - 2 = 25$
C: $35 - 8 - 11 - 2 = 14$
Finally, add all the numbers shown in the circles and subtract from 100 to find

$$100 - 27 - 13 - 11 - 8 - 25 - 2 - 14 = 0$$

52. Let C = {like comedies}; V = {like variety}; D = {like drama}

Draw the Venn diagram.



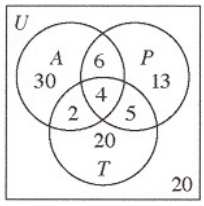
We now add all of the numbers in the Venn diagram:

$$32 + 11 + 15 + 9 + 7 + 5 + 21 + 2 = 102$$

No, there were 102 persons polled, not 100, or perhaps he made up data.

53. Let A = {drive alone}; P = {carpool}; T = {public transportation}

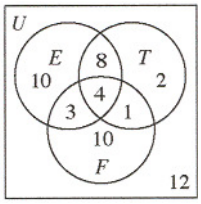
Draw the Venn diagram.



20 used none of the above means of transportation.

54. a. Let E = {favor Prop. 8}; T = {favor Prop. 13}; F = {favor Prop. 5}

Draw the Venn diagram.

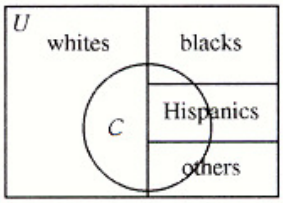


b. Look at the Venn diagram to see it is 12.

c. "And" means both E and F, so we find $3 + 4 = 7$.

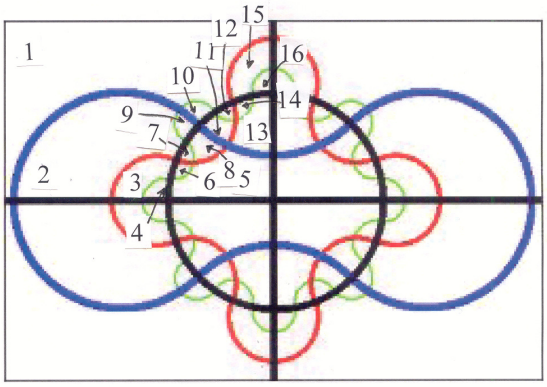
Level 3 Problem Solving, page 67

55. a. Let U = {people who live in the United States}
and C = {people who use computers}



b. Cannot add percentages. Suppose, $|W| = 220$, $|B| = 33$, $|H| = 25$ so $0.27|W| \approx 60$, $0.14|B| \approx 4.6$, and $0.13|H| \approx 3.25$. Thus, among the blacks and Hispanics, $4.6 + 3.25 \approx 8$ million who use computers does not even come close to the estimated 60 million white people who use the computer.

56.



We have labeled the 16 regions in one quarter of the Venn diagram; the numbering would be identical in the other three quarters. For example, region 1 is outside all four sets; region 2 is inside the blue set, but outside all the others, and so forth.

57. Answers vary.

a. Inside *E* but outside *A, B, C,* and *D;* $\overline{(A \cup B \cup C \cup D)} \cap \text{E}$

b. Inside *A* and *E* but outside *B, C,* and *D;* $(A \cap E) \cap \overline{(B \cup C \cup D)}$

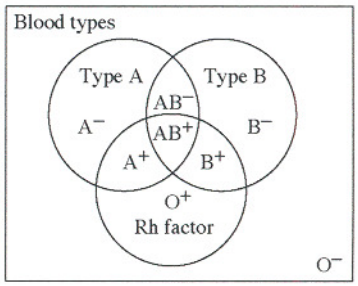
c. Inside *B, E,* and *D* but outside *A* and *C*; $(B \cap E \cap D) \cap \overline{(A \cup C)}$

d. Inside all sets; $A \cap B \cap C \cap D \cap E$

e. Inside *A, B* and *E* but outside *C* and *D;* $(A \cap B \cap E) \cap \overline{(C \cup D)}$

58. region 32 **59**. region 2

60.



2.4 Finite and Infinite Sets, page 68

New Terms Introduced in this Section

Cartesian product	Countable set	Countably infinite
Finite set	Fundamental counting principle	Infinite set
One-to-one correspondence	Proof by contradiction	Uncountable set
Uncountably infinite		

Teaching Suggestions

I ask the class "What is the largest number whose name you know?" and that discussion evolves from the finite to the infinite. In class I attempt to lead the class to a definition of what we mean by "infinite." When we discuss the fact that the counting numbers and integers are infinite and countable, the students are ready to agree because that matches their intuition. Next, Example 3, page 70, is believable to them, but they need to have a classroom demonstration. On the other hand, when we discuss Example 4, page 70, and show that the set of real numbers is uncountable, and then say that this is *another* order of infinity, there is total disbelief.

Level 1, page 73

1. Answers vary. It is a very powerful method for counting very large and complex sets.

2. Answers vary. **3.** The Cartesian product of set A and B, denoted by $A \times B$, is the set of all *ordered pairs* (x, y) where $x \in A$ and $y \in B$. **4.** Answers vary. The cardinality of a Cartesian product can be found by using the fundamental counting principle.

5. $A \times B = \{(c, w), (c, x), (d, w), (d, x), (f, w), (f, x)\}$

6. $B \times A = \{(w, c), (x, c), (w, d), (x, d), (w, f), (x, f)\}$

7. $\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c), (5, a), (5, b), (5, c)\}$

8. $\{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2), (a, 3), (b, 3), (c, 3), (a, 4), (b, 4), (c, 4), (a, 5), (b, 5), (c, 5)\}$

9. $\{(2, a), (2, e), (2, i), (2, o), (2, u), (3, a), (3, e), (3, i), (3, o), (3, u), (4, a), (4, e), (4, i), (4, o), (4, u)\}$

10. $\{(a, 2), (a, 3), (a, 4), (e, 2), (e, 3), (e, 4), (i, 2), (i, 3), (i, 4), (o, 2), (o, 3), (o, 4), (u, 2), (u, 3), (u, 4)\}$

11. $(190 - 48) + 1 = 143$ **12.** $\left(\frac{404-16}{4}\right) + 1 = 98$ **13.** $42 + 42 + 1 = 85$

14. $998 + 86 + 1 = 1{,}085$ **15.** $\aleph_0$ **16.** $\aleph_0$ **17.** $20 \times 23 = 460$

18. $50 \times 3 = 150$ **19.** $\aleph_0$ **20.** $\aleph_0$

Level 2, page 73

21. **a.** 0 **b.** $|\varnothing| = 0$ **22**. **a.** 1 **b.** $|\{\varnothing\}| = 1$ **23**. **a.** 1 **b.** $|\{0\}| = 1$ **24**. **a.** 26
b. 50 **c.** $S = |\{A, B\}| = 2$; the set S has two elements, namely the sets A and B.
25. $|A| = 26$ and $|B| = 50$, so $|A \times B| = 26 \times 50 = 1{,}300$ **26.** $|A \times A| = 26 \times 26 = 676$
27. $|C \times D| = 100 \times 50 = 5{,}000$ **28.** $|M \times N| = 8 \times 8 = 64$
29. $\{m, a, t\}$; $\{m, a, t\}$; there are others.
$\{1, 2, 3\}$; $\{3, 2, 1\}$

30. $\{1, 2, 3, \cdots, n, n+1, \cdots, 999, 1000\}$
$\{7964, 7965, 7966, \cdots, n+7963, n+7964, \cdots, 8962, 8963\}$
Thus, the sets have the same cardinality.

31. $\{1, 2, 3, \cdots, n, n+1, \cdots, 353, 354, 355, 356, \cdots 586, 587\}$
$\{550, 551, 552, \cdots n+549, n+550, \cdots 902, 903\}$
Thus, the sets do not have the same cardinality.

32. $\{48, 49, 50, \cdots, n, \cdots, 783, 784, ???\}$
$\{485, 487, 489, \cdots, 2n+389, \cdots, 1955, 1957, 1959, \cdots\}$
Not one-to-one, so they do not have the same cardinality.

33. **a**. finite **b**. infinite **34.** **a**. finite **b**. finite

Level 3, page 74

35. $\{1, 2, 3, \cdots, n, \cdots\}$
$\{-1, -2, -3, \cdots, -n, \cdots\}$
Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

36. $\{1, 2, 3, \cdots, n, \cdots\}$
$\{1000, 3000, 5000, \cdots, 1000(2n-1). \cdots\}$
Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

37. $\{1, 2, 3, \cdots, n, \cdots\}$
$\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, \cdots\}$
Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

38. $\{2, 4, 6, \cdots, 2n, \cdots\}$
$\updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow$
$\{1, 2, 3, \cdots, n, \cdots\}$

Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

39. $\mathbb{W} = \{0, 1, 2, 3, \cdots, n, \cdots\}$
$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow$
$\{1, 2, 3, 4, \cdots \quad n+1, \cdots\}$

Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

40. $\mathbb{Z} = \{0, 1, -1, 2, -3, \cdots, n, -n, \cdots\}$
$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow \quad \updownarrow$
$\{1, 2, 3, 4, 5, \cdots \quad 2n, \quad 2n+1, \cdots\}$

Let each negative number "line up" after its positive counterpart. Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

41. $\{1, 3, 5, \cdots, 2n-1, \cdots\}$
$\updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow$
$\{1, 2, 3, \cdots, n, \cdots\}$

Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

42. $\{5, 10, 15, \cdots, 5n, \cdots\}$
$\updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow$
$\{1, 2, 3, \cdots, n, \cdots\}$

Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

43. $\{1, 2, 3, \cdots, n, \cdots\}$
$\updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow$
$\{1, 3, 9, \cdots \quad 3^{n-1}, \cdots\}$

Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

44. $\{1, 2, 3, \cdots, n, \cdots\}$
$\updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow$
$\{-85, -80, -75, \cdots, 100-5(n+2), \cdots\}$

Since these sets can be put into a 1-1 correspondence, they have the same cardinality; namely, $\aleph_0$.

45. $\mathbb{W} = \{0, 1, 2, 3, \cdots, n, \cdots\}$
$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \qquad \updownarrow$
$\{1, 2, 3, 4, \cdots \quad n+1, \cdots\}$

Since $\mathbb{W}$ can be put into a one-to-one correspondence with a proper subset of itself, it is infinite.

46. $\mathbb{N} = \{1, 2, 3, 4, \cdots, n, \cdots\}$
$\{2, 3, 4, 5, \cdots, n+1, \cdots\}$
Since $\mathbb{N}$ can be put into a one-to-one correspondence with a proper subset of itself, it is infinite.

47. $\{12, 14, 16, \cdots, n, \cdots\}$ Thus, this set is infinite.
$\{14, 16, 18, \cdots, n+2, \cdots\}$

48. $\{4, 44, 444, \cdots,$ integer with n 4s, $\cdots\}$ Thus, this set is infinite.
$\{44, 444, 4444, \cdots,$ integer with $n+1$ 4s, $\cdots\}$

49. $\mathbb{W} = \{0, 1, 2, 3, \cdots, n, \cdots\}$ Thus, $\mathbb{W}$ is countably infinite.
$\{1, 2, 3, 4, \cdots, n+1, \cdots\}$

50. $\mathbb{Z} = \{0, 1, -1, 2, -3, \cdots, n, -n, \cdots\}$ Thus, $\mathbb{Z}$ is countably infinite.
$\{1, 2, 3, 4, 5, \cdots, 2n, 2n+1, \cdots\}$

51. $\mathbb{Q}$ is countably infinite because it can be put into a one-to-one correspondence with the counting numbers (see Example 3).

52. $\mathbb{R}$ is uncountably infinite because it cannot be put into a one-to-one correspondence with the counting numbers (see Example 4).

53. $\{2, 4, 8, \cdots, 2^n, \cdots\}$
$\{1, 2, 3, \cdots, n, \cdots\}$
Thus, the set $\{2, 4, 8, 16, 32, \cdots\}$ is countably infinite.

54. $\{1, 4, 9, \cdots, n^2, \cdots\}$
$\{1, 2, 3, \cdots, n, \cdots\}$
Thus, the set $\{1, 4, 9, 16, 25, \cdots\}$ is countably infinite.

55. If a set is uncountable, then is infinite, by definition; the statement is true. **56.** Some infinite sets are countable and some are uncountable, so the statement is false. **57.** Some infinite sets have cardinality $\aleph_0$, while other infinite sets have larger cardinality; the statement is false. **58.** Of course not, $\{1\}$ and $\{4\}$ are equivalent, but certainly not infinite, so the statement is false.

Level 3 Problem Solving, page 74

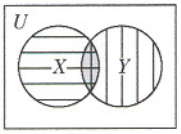
59. Answers vary; the set, *E*, of even integers has cardinality $\aleph_0$; the set, *O*, of odd integers has cardinality $\aleph_0$. If we add the cardinality of the even integers to the cardinality of the odd integers, we have $\aleph_0 + \aleph_0$. However, if we put the even integers together with the odd

integers, we have the set of counting numbers, which has cardinality $\aleph_0$. Thus, our example illustrates why $\aleph_0 + \aleph_0 = \aleph_0$. **60.** All are equal.

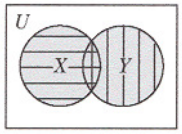
Chapter 2 Review Questions, page 76

1. a. $\{1, 2, 3, 4, 5, 6, 7, 9, 10\}$ **b.** $\{9\}$ **c.** $\{2, 4, 6, 8, 10\}$ **d.** $\{1, 3, 5, 7, 8\}$
2. a. 0 **b.** 10 **c.** 5 **d.** 5 **3. a.** 1 **b.** 25 **c.** 50 **d.** 25 **4. a.** $\{1, 2, 3, 4, 5, 6, 7, 8, 10\}$
b. $\{1, 2, 3, 4, 5, 6, 7, 8, 10\}$ **c.** $\varnothing$ **d.** Yes, they are equal from De Morgan's law.
5. a. $\{2, 4, 6, 8, 10\}$ **b.** $\{2, 4, 6, 10\}$ **6. a.** $\subset$ or $\subseteq$ **b.** $\in$ **c.** $=$ or $\subseteq$ **d.** $=$ or $\subseteq$
e. $\subset$ or $\subseteq$ **7. a.** 25 **b.** 41 **c.** 20 **d.** 46

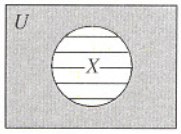
8. a.



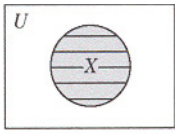
b.



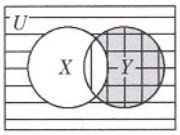
9. a.



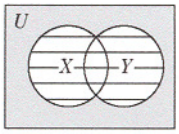
b.



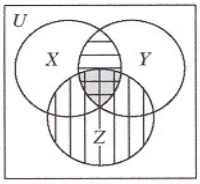
10. a.



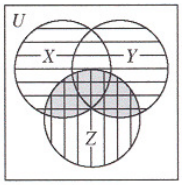
b.



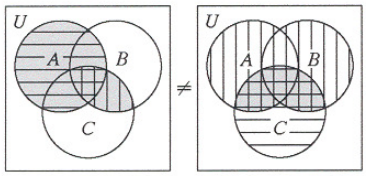
11. a.



b.

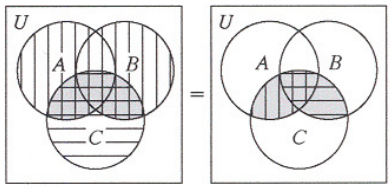


12.



The statement has been disproved.

13.



The statement has been proved.

14. a. The set of rational numbers is the set of all numbers of the form $\frac{a}{b}$ such that a is an integer and b is a counting number.

b. $\frac{2}{3}$; answers vary.

15. $\{5, 10, 15, \cdots, 5n, \cdots\}$

$\{10, 20, 30, \cdots, 10n, \cdots\}$

Since the second set is a proper subset of the first set, we see that the set F is infinite.

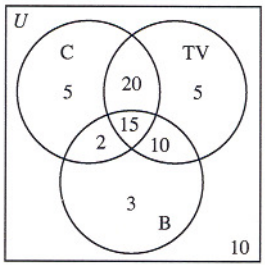
16. a. $\{n \mid (-n) \in \mathbb{N}\}$; answers vary.

b. $\{1, 2, 3, \cdots, n, \cdots\}$

$\{-1, -2, -3, \cdots, -n, \cdots\}$

Since the first set is the set of counting numbers, it has cardinality $\aleph_0$; so the given set also has cardinality $\aleph_0$ since it can be put into a one-to-one correspondence with the set of counting numbers.

17.



Ten students had none of the items.

18. $A^-, AB^-, B^-, A^+, AB^+, B^+, O^+, O^-$

19. $A^-, AB^-, B^-, A^+, AB^+, B^+, O^-$

20. A, B, O^+

Group Research Projects, page 77

Generally, complete answers to these problems will not be given in this *Instructor's Manual*. But notice that these problems introduce the student to some of the more important literature in mathematics. Some good general references, in addition to those listed as references in this section, are:

Mathematics, Life Science Library. This is an excellent book to introduce mathematics to the layperson.

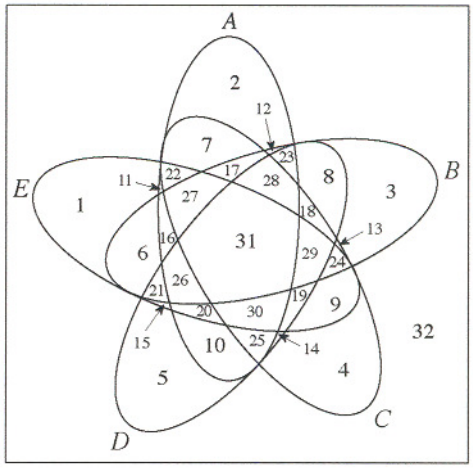
The World of Mathematics by Newman. This classic is full of ideas.

Mathematics and the Imagination by Kasner and Newman.

These three books should be on the shelf of every instructor teaching this course. They will provide endless ideas for discussion. These problems will also introduce the students to some of the mathematical journals: *The Arithmetic Teacher, The Mathematics Teacher*, and *The Journal of Recreational Mathematics.* There are a multitude of links for research projects at the website for this book, **www.mathnature.com**.

G3. 17% (Solution by using a Venn diagram.)

G4. Answers vary. Consider the following Venn diagram.



I: Inside *E* but outside *A, B, C,* and *D;* $\overline{(A \cup B \cup C \cup D)} \cap E$

II: Inside A but outside $B, C, D,$ and E; $A \cap \overline{B \cup C \cup D \cup E}$

III: Inside B but outside $A, C, D,$ and E; $B \cap \overline{A \cup C \cup D \cup E}$

IV: Inside C but outside $A, B, D,$ and E; $C \cap \overline{A \cup B \cup D \cup E}$

V: Inside D but outside $A, B, C,$ and E; $D \cap \overline{A \cup B \cup C \cup E}$

VI: Inside B and *E* but outside A, *C,* and *D;* $(B \cap E) \cap \overline{(A \cup C \cup D)}$

VII: Inside *A* and C but outside *B,* $D,$ and $E;$ $(A \cap C) \cap \overline{(B \cup D \cup E)}$

VIII: Inside B and D but outside A, *C,* and $E;$ $(B \cap D) \cap \overline{(A \cup C \cup E)}$

IX: Inside C and *E* but outside A, *B,* and *D;* $(C \cap E) \cap \overline{(A \cup B \cup D)}$

X: Inside *A* and D but outside *B, C,* and $E;$ $(A \cap D) \cap \overline{(B \cup C \cup E)}$

XI: Inside *A* and *E* but outside *B, C,* and *D;* $(A \cap E) \cap \overline{(B \cup C \cup D)}$

XII: Inside *A* and B but outside $C,$ $D,$ and $E;$ $(A \cap B) \cap \overline{(C \cup D \cup E)}$

XIII: Inside B and C but outside A, D, and E; $(B \cap C) \cap \overline{(A \cup D \cup E)}$
XIV: Inside C and D but outside A, B, and E; $(C \cap D) \cap \overline{(A \cup B \cup E)}$
XV: Inside D and E but outside A, B, and C; $(D \cap E) \cap \overline{(A \cup B \cup C)}$
XVI: Inside A, B and E but outside C and D; $(A \cap B \cap E) \cap \overline{(C \cup D)}$
XVII: Inside A, B and C but outside D and E; $(A \cap B \cap C) \cap \overline{(D \cup E)}$
XVIII: Inside C, B and D but outside A and E; $(C \cap B \cap D) \cap \overline{(A \cup E)}$
XIX: Inside C, D and E but outside A and B; $(C \cap D \cap E) \cap \overline{(A \cup B)}$
XX: Inside A, D and E but outside B and C; $(A \cap D \cap E) \cap \overline{(B \cup C)}$
XXI: Inside B, E, and D but outside A and C; $(B \cap E \cap D) \cap \overline{(A \cup C)}$
XXII: Inside A, C and E but outside B and D; $(A \cap C \cap E) \cap \overline{(B \cup D)}$
XXIII: Inside A, B and D but outside C and E; $(A \cap B \cap D) \cap \overline{(C \cup E)}$
XXIV: Inside B, C and E but outside A and D; $(B \cap C \cap E) \cap \overline{(A \cup D)}$
XXV: Inside A, C and D but outside B and E; $(A \cap C \cap D) \cap \overline{(B \cup E)}$
XXVI: Outside C and inside A, B, D and E; $A \cap B \cap \overline{C} \cap D \cap E$
XXVII: Outside D and inside A, B, C and E; $A \cap B \cap C \cap \overline{D} \cap E$
XXVIII: Outside E and inside A, B, C and D; $A \cap B \cap C \cap D \cap \overline{E}$
XXIX: Outside A and inside B, C, D and E; $\overline{A} \cap B \cap C \cap D \cap E$
XXX: Outside B and inside A, C, D and E; $A \cap \overline{B} \cap C \cap D \cap E$
XXXI: Inside all sets; $A \cap B \cap C \cap D \cap E$
XXXII: Outside all sets: $\overline{A \cup B \cup C \cup D \cup E}$

G5. Answers vary. The eight sets are:
$A = \{\text{council of Europe}\}$; $|A = 48|$
$B = \{\text{European Economic Area}\}$; $|B| = 30$
$C = \{\text{European Union}\}$; $|C| = 27$
$D = \{\text{Eurozone}\}$; $|D| = 17$
$E = \{\text{EU Customs Union}\}$; $|E| = 31$
$F = \{\text{Schengen Area}\}$; $|F| = 26$
$G = \{\text{European Free Trade Association}\}$; $|G| = 4$
$H = \{\text{Agreement with EU to mint Euros}\} = |H| = 3$
The total number of regions formed by eight sets is 256. The entity at the lower right green circle is on one, Switzerland is in two, Liechtenstein is in three, Great Britain is in four, Ireland (green/white/orange) is in five, and Italy (green/white/red) is in six. None are in seven or eight of the regions.

G6. A *paradox* is a statement that is inescapable, but yet impossible. Consider the *barber's rule*. If he does shave himself, then according to the barber's rule, he does not shave himself. On the other hand, if he does not shave himself, then, according to the barber's rule, he shaves himself. We can only conclude that there can be no such barber's rule. But why not? The paper you should write will address this question. The *barber's rule* is better known as the famous Russell paradox.

Individual Research Projects, page 78

P2.1 Begin with the single cell cages, then move to the double cages until you have a solution.

80× 5	4	3 3	5− 1	6	2÷ 2
4	11+ 6	5	1− 2	3	1
9× 3	2 2	3− 1	4	30× 5	6
1	3	11+ 6	5	2÷ 2	4
6 6	8× 1	2	13+ 3	4	8+ 5
10× 2	5	4	6	1 1	3

P2.2 We will use Pólya's method to answer this question.

Understand the Problem. Imagine a long list of the numbers: 1, 2, 3,⋯, 999,999, 1,000,000, 1,000,001, 1,000,002. Suppose we asked for the millionth positive integer on this list — that's easy. It is 1,000,000. Do you understand what is meant by "perfect squares" and "perfect cubes"?

Perfect squares: $1^2 = 1$; $2^2 = 4$, $3^2 = 9$; $4^2 = 16$; 25; 36; 49; 64; 81; ⋯

Perfect cubes: $1^3 = 1$; $2^3 = 8$; $3^3 = 27$; $4^3 = 64$; 125; 216; 343; 512; ⋯

Now, suppose we cross out the perfect squares; for each number crossed off, the millionth number in the list changes. That is, cross off 1 and the millionth number is 1,000,001; cross off 1 and 4 and the millionth number is 1,000,002. In this problem, we need to cross out all the perfect squares *and* perfect cubes. *After* we have done this, we look at the list and find the millionth number.

Devise a Plan. It should be clear that we need to know how many numbers are crossed out.

Let $U = \{1, 2, 3, \cdots, 1{,}000{,}000\}$;

$S = \{\text{perfect squares}\}$ and

$C = \{\text{perfect cubes}\}$.

We wish to find $|S|$: We know that $1{,}000{,}000 = (10^3)^2$ so there are $10^3 = 1{,}000$ perfect squares less than or equal to 1,000,000. Therefore, $|S| = 1{,}000$. Next, find $|C|$: We also know that $1{,}000{,}000 = (10^2)^3$ so there are $10^2 = 100$ perfect cubes less than or equal to 1,000,000. Therefore, $|C| = 100$.

If you think about crossing out all of the perfect squares and then all of the perfect cubes, you must notice that some numbers are on both lists. That is, what numbers are in the set $S \cap C$? These are the sixth powers:

$S \cap C$: $1^6 = 1$; $2^6 = 64$; $3^6 = 729$; $4^6 = 4{,}096$; $\cdots$ $10^6 = 1{,}000{,}000$.

There are 10 perfect sixth powers, so $|S \cap C| = 10$.

Carry Out the Plan. We use the formula from this chapter to find $|S \cup C|$:

$$|S \cup C| = |S| + |C| - |S \cap C| = 1{,}000 + 100 - 10 = 1{,}090$$

If you cross out 1,090 numbers then the millionth number not crossed out is 1,001,090.

Look Back. The answer 1,001,090 is not necessarily correct because we crossed out perfect squares, cubes, and sixth powers up to 1,000,000. But each time we crossed out a number, the end number was extended by 1. Thus we need to know whether there are any perfect squares, cubes, or sixth powers between 1,000,000 and 1,001,090 (the target range). Let's check the next number on each of our lists:

Perfect squares:	$1{,}001^2 = 1{,}002{,}001$, so it is not in the target range;
Perfect cubes:	$101^3 = 1{,}030{,}301$, so it is not in the target range;
Perfect sixth powers:	$11^6 = 1{,}771{,}561$, so it also is not in the target range.

The millionth number that is not a perfect square or perfect cube is 1,001,090.

P2.3 We continue with the process began in the solution of the previous project and use the formula

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Let $U = \{1, 2, 3, \cdots\}$,
$S = \{\text{perfect squares}\}$, $|S| = 1{,}000$;
$C = \{\text{perfect cubes}\}$ $|C| = 100$;
$F = \{\text{perfect fifth powers}\}$, $|F| = 15$ since $16^5 > 1{,}000{,}000$.

The cardinality of the set of numbers that are both perfect squares and perfect cubes is $|S \cap C| = 10$.

$S \cap F = \{\text{perfect tenth powers}\} = \{1^{10}, 2^{10}, 3^{10}\}$, so $|S \cap F| = 3$
$C \cap F = \{\text{perfect fifteenth powers}\} = \{1^{15}, 2^{15}\}$, so $|C \cap F| = 2$
$S \cap C \cap F = \{\text{perfect thirtieth powers}\} = \{1^{30}\}$, so $|S \cap C \cap F| = 1$

$$\begin{aligned} |S \cup C \cup F| &= |S| + |C| + |F| - |S \cap C| - |S \cap F| - |C \cap F| + |S \cap C \cap F| \\ &= 1{,}000 + 100 + 15 - 10 - 3 - 2 + 1 \\ &= 1{,}101 \end{aligned}$$

Thus, the millionth positive integer that is *not* a perfect square, cube, or fifth power is 1,001,101.

P2.4 Answers vary. You should write at least one page and should also show your sources.

P2.5 We pattern our demonstration after Example 4, page 70. Let's suppose that $\mathbb{S}$ is countable. Then, there is some one-to-one correspondence between $\mathbb{S}$ and $\mathbb{N}$, say:

$$\begin{aligned} 1 &\leftrightarrow 0.285938\cdots \\ 2 &\leftrightarrow 0.444444\cdots \\ 3 &\leftrightarrow 0.500000\cdots \\ 4 &\leftrightarrow 0.123456\cdots \\ &\vdots \end{aligned}$$

Now, if we assume that there is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{S}$ then *every* decimal number is in the above list. To show this is not possible, we construct a new decimal as follows. The first digit of this new decimal is *any* digit different from the first digit of the entry corresponding to the first correspondence. (That is, anything other than 2 using the above listed correspondence). The second digit is any digit different from the second digit of the entry corresponding to the second correspondence (4 in this example). Do the same for *all* the numbers in the one-to-one correspondence. Because of the way we have constructed this new number, it is not on the list. But we began by assuming that all numbers were on the list (*i.e.,* part of the one-to-one correspondence). Since both these statements cannot be true, the original assumption has led to a contradiction. This forces us to accept the only possible alternative to the original assumption. That is, it is not possible to set up a one-to-one correspondence between $\mathbb{S}$ and $\mathbb{N}$, which means that $\mathbb{S}$ is uncountable.