CHAPTER 2

DTMCs: Transient Behavior

Modeling Exercises

2.1. The state space of $\{X_n, n \geq 0\}$ is $S = \{0, 1, 2, 3, ...\}$. Suppose $X_n = i$. Then the age of the lightbulb in place at time n is i. If this light bulb does not fail at time $n+1$, then $X_{n+1} = i+1$. If it fails at time $n+1$, then a new lightbulb is put in at time $n+1$ with age 0, making $X_{n+1} = 0$. Let Z be the lifetime of a lightbulb. We have

$$\begin{aligned}
\mathsf{P}(X_{n+1} = 0 | X_n = i, X_{n-1}, ..., X_0) &= \mathsf{P}(\text{lightbulb of age } i \text{ fails at age } i+1) \\
&= \mathsf{P}(Z = i+1 | Z > i) \\
&= \frac{p_{i+1}}{\sum_{j=i+1}^{\infty} p_j}
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathsf{P}(X_{n+1} = 0 | X_n = i, X_{n-1}, ..., X_0) &= \mathsf{P}(Z > i+1 | Z > i) \\
&= \frac{\sum_{j=i+2}^{\infty} p_j}{\sum_{j=i+1}^{\infty} p_j}
\end{aligned}$$

It follows that $\{X_n, n \geq 0\}$ is a success-runs DTMC with

$$p_i = \frac{\sum_{j=i+2}^{\infty} p_j}{\sum_{j=i+1}^{\infty} p_j},$$

and

$$q_i = \frac{p_{i+1}}{\sum_{j=i+1}^{\infty} p_j},$$

for $i \in S$.

2.2 The state space of $\{Y_n, n \geq 0\}$ is $S = \{1, 2, 3, ...\}$. Suppose $Y_n = i > 1$, then the remaining life decreases by one at time $n+1$. Thus $X_{n+1} = i-1$. If $Y_n = 1$, a new light bulb is put in place at time $n+1$, thus Y_{n+1} is the lifetime of the new light bulb. Let Z be the lifetime of a light bulb. We have

$$\mathsf{P}(Y_{n+1} = i-1 | X_n = i, X_{n-1}, ..., X_0) = 1, \quad i \geq 2,$$

and

$$\mathsf{P}(X_{n+1} = k|X_n = 1, X_{n-1}, ..., X_0) = \mathsf{P}(Z = k) = p_k, \quad k \geq 1.$$

2.3. Initially the urn has $w + b$ balls. At each stage the number of balls in the urn increases by $k - 1$. Hence after n stages, the urn has $w + b + n(k - 1)$ balls. X_n of them are black, and the remaining are white. Hence the probability of getting a black ball on the $n + 1$st draw is

$$\frac{X_n}{w + b + n(k-1)}.$$

If the $n + 1$st draw is black, $X_{n+1} = X_n + k - 1$, and if it is white, $X_{n+1} = X_n$. Hence

$$\mathsf{P}(X_{n+1} = i|X_n = i) = 1 - \frac{i}{w + b + n(k-1)},$$

and

$$\mathsf{P}(X_{n+1} = i + k - 1|X_n = i) = \frac{i}{w + b + n(k-1)}.$$

Thus $\{X_n, n \geq 0\}$ is a DTMC, but it is not time homogeneous.

2.4. $\{X_n, n \geq 0\}$ is a DTMC with state space $\{0 = \text{dead}, 1 = \text{alive}\}$ because the movements of the cat and the mouse are independent of the past while the mouse is alive. Once the mouse is dead, it stays dead. If the mouse is still alive at time n, he dies at time $n + 1$ if both the cat and mouse choose the same node to visit at time $n + 1$. There are $N - 2$ ways for this to happen. In total there are $(N - 1)^2$ possible ways for the cat and the mouse to choose the new nodes. Hence

$$\mathsf{P}(X_{n+1} = 0|X_n = 1) = \frac{N-2}{(N-1)^2}.$$

Hence the transition probability matrix is given by

$$P = \begin{bmatrix} 1 & 0 \\ \frac{N-2}{(N-1)^2} & 1 - \frac{N-2}{(N-1)^2} \end{bmatrix}.$$

2.5. Let $X_n = 1$ if the weather is sunny on day n, and 2 if it is rainy on day n. Let $Y_n = (X_{n-1}, X_n)$, be the vector of weather on day $n - 1$ and $n, n \geq 1$. Now suppose $Y_n = (1, 1)$. This means the weather was sunny on day $n - 1$ and n. Then, it will be sunny on day $n + 1$ with probability .8 and the new weather vector will be $Y_{n+1} = (1, 1)$. On the other hand it will rain on day $n + 1$ with probability .2, and the weather vector will be $Y_{n+1} = (1, 2)$. These probabilities do not depend on the weather up to time $n - 2$, i.e., they are independent of $Y_1, Y_2, ...Y_{n-2}$. Similar analysis in other states of Y_n shows that $\{Y_n, \ n \geq 1\}$ is a DTMC on state space $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$ with the following transition probability matrix:

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ .75 & .25 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}.$$

2.6. The state space is $S = \{0, 1, \cdots, K\}$. Let

$$\alpha_i = \binom{K}{i} p^i (1-p)^{K-i}, \quad 0 \le i \le K.$$

Thus, when a functioning system fails, i components fail simultaneously with probability α_i, $i \ge 1$. The $\{X_n, n \ge 0\}$ is a DTMC with transition probabilities:

$$p_{0,i} = \alpha_i, \quad 0 \le i \le K,$$
$$p_{i,i-1} = 1, \quad 1 \le i \le K.$$

2.7. Suppose $X_n = i$. Then, $X_{n+1} = i + 1$ if the first coin shows heads, while the second shows tails, which will happen with probability $p_1(1 - p_2)$, independent of the past. Similarly, $X_{n+1} = i - 1$ if the first coin shows tails and the second coin shows heads, which will happen with probability $p_2(1-p_1)$, independent of the past. If both coins show heads, or both show tails, $X_{n+1} = i$. Hence, $\{X_n, n \ge 0\}$ is a space homogeneous random walk on $S = \{..., -2, -1, 0, 1, 2, ...\}$ (see Example 2.5) with

$$p_i = p_1(1 - p_2), \quad q_i = p_2(1 - p_1), \quad r_i = 1 - p_i - q_i.$$

2.8. We define X_n, the state of the weather system on the nth day, as the length of the current sunny or rainy spell. The state is k, $(k = 1, 2, 3, ...)$, if the weather is sunny and this is the kth day of the current sunny spell. The state is $-k$, $(k = 1, 2, 3, ..)$, if the the weather is rainy and this is the kth day of the current rainy spell. Thus the state space is $S = \{\pm 1, \pm 2, \pm 3, ...\}$.

Now suppose $X_n = k$, $(k = 1, 2, 3, ...)$. If the sunny spell continues for one more day, then $X_{n+1} = k + 1$, or else a rainy spell starts, and $X_{n+1} = -(k + 1)$. Similarly, suppose $X_n = -k$. If the rainy spell continues for one more day, then $X_{n+1} = -(k+1)$, or else a sunny spell starts, and $X_{n+1} = 1$. The Markov property follows from the fact that the lengths of the sunny and rainy spells are independent. Hence, for $k = 1, 2, 3, ...$,

$$\begin{aligned}
\mathsf{P}(X_{n+1} = k + 1 | X_n = k) &= p_k, \\
\mathsf{P}(X_{n+1} = -1 | X_n = k) &= 1 - p_k, \\
\mathsf{P}(X_{n+1} = -(k + 1) | X_n = -k) &= q_k, \\
\mathsf{P}(X_{n+1} = 1 | X_n = -k) &= 1 - q_k.
\end{aligned}$$

2.9. Y_n is the outcome of the nth toss of a six sided fair die. $S_n = Y_1 + ...Y_n$. $X_n = S_n$ (mod 7). Hence we see that

$$X_{n+1} = X_n + Y_{n+1} (\text{mod } 7).$$

Since Y_n s are iid, the above equation implies that $\{X_n, n \geq 0\}$ is a DTMC with state space $S = \{0, 1, 2, 3, 4, 5, 6\}$. Now, for $i, j \in S$, we have

$$\begin{aligned} \mathsf{P}(X_{n+1} = j | X_n = i) &= \mathsf{P}(X_n + Y_{n+1} (\text{mod } 7) = j | X_n = i) \\ &= \mathsf{P}(i + Y_{n+1} (\text{mod } 7) = j) \\ &= \begin{cases} 0 & \text{if } i = j \\ \frac{1}{6} & \text{if } i \neq j. \end{cases} \end{aligned}$$

Thus the transition probability matrix is given by

$$P = \begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \end{bmatrix}.$$

2.10. State space of $\{X_n, n \geq 0\}$ is $S = \{0, 1, \cdots, r-1\}$. We have

$$X_{n+1} = X_n + Y_{n+1} (\text{mod r}),$$

which shows that $\{X_n, n \geq 0\}$ is a DTMC. We have

$$\mathsf{P}(X_{n+1} = j | X_n = i) = \mathsf{P}(Y_{n+1} = (j - i)(\text{mod r})) = \sum_{m=0}^{\infty} \alpha_{j-i+mr}.$$

Here we assume that $\alpha_k = 0$ for $k \leq 0$.

2.11. Let B_n (G_n) be the bar the boy (girl) is in on the nth night. Then $\{(B_n, G_n), n \geq 0\}$ is a DTMC on $S = \{(1,1), (1,2), (2,1), (2,2)\}$ with the following transition probability matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a(1-d) & ad & (1-a)(1-d) & (1-a)d \\ (1-b)c & (1-b)(1-c) & bc & b(1-c) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The story ends in bar k if the bivariate DTMC gets absorbed in state (k, k), for $k = 1, 2$.

2.12. Let Q be the transition probability matrix of $\{Y_n, n \geq 0\}$. Suppose $Z_m = f(i)$, that the DTMC Y is in state i when the filled gas for the mth time. Then, the student fills gas next after $11 - i$ days. The DTMC Y will be in state j at that time with probability $[Q^{11-i}]_{ij}$. This shows that $\{Z_m, m \geq 0\}$ is a DTMC with state space $\{f(0), f(1), \cdots, f(10)\}$, with transition probabilities

$$\mathsf{P}(Z_{m+1} = f(j) | Z_m = f(i)) = [Q^{11-i}]_{ij}.$$

2.13. Following the analysis in Example 2.1b, we see that $\{X_n, n \geq 0\}$ is a DTMC on state space $S = \{1, 2, 3, ..., k\}$ with the following transition probabilities:

$$\mathsf{P}(X_{n+1} = i | X_n = i) = p_i, \quad 1 \leq i \leq k,$$

$$\mathsf{P}(X_{n+1} = i + 1 | X_n = i) = 1 - p_i, \quad 1 \leq i \leq k - 1,$$

$$\mathsf{P}(X_{n+1} = 1 | X_n = k) = 1 - p_k.$$

2.14. Let the state space be $\{0, 1, 2, 12\}$, where the state is 0 if both components are working, 1 if component 1 alone is down, 2 i f component 2 alone is down, and 12 if components 1 and 2 are down. Let X_n be the state on day n. $\{X_n, n \geq 0\}$ is a DTMC on $\{0, 1, 2, 12\}$ with tr pr matrix

$$P = \begin{bmatrix} \alpha_0 & \alpha_1 \alpha_2 & \alpha_{12} & \\ r_1 & 1 - r_1 & 0 & 0 \\ r_2 & 0 & 1 - r_2 & 0 \\ 0 & 0 & r_1 & 1 - r_1 \end{bmatrix}.$$

Here we have assumed that if both components fail, we repair component 1 first, and then component 2.

2.15. Let X_n be the pair that played the nth game. Then $X_0 = (1, 2)$. Suppose $X_n = (1, 2)$. Then the nth game is played between player 1 and 2. With probability b_{12} player 1 wins the game, and the next game is played between players 1 and 3, thus making $X_{n+1} = (1, 3)$. On the other hand, player 2 wins the game with probability b_{21}, and the next game is played between players 2 and 3, thus making $X_{n+1} = (2, 3)$. Since the probabilities of winning are independent of the past, it is clear that $\{X_n, n \geq 0\}$ is a DTMC on state space $\{(1, 2), (2, 3), (1, 3)\}$. Using the arguments as above, we see that the transition probabilities are given by

$$P = \begin{bmatrix} 0 & b_{21} & b_{12} \\ b_{23} & 0 & b_{32} \\ b_{13} & b_{31} & 0 \end{bmatrix}.$$

2.16. Let X_n be the number of beers at home when Mr. Al Anon goes to the store. Then $\{(X_n, Y_n), n \geq 0\}$ is DTMC on state space

$$S = \{(0, L), (1, L), (2, L), (3, L), (4, L), (0, H), (1, H), (2, H), (3, H), (4, H)\}$$

with the following transition probability matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1-\beta & 0 & 0 & 0 & 0 & \beta & 0 \end{bmatrix}.$$

2.17. We see that

$$X_{n+1} = \max\{X_n, Y_{n+1}\}.$$

Since the Y_n's are iid, $\{X_n, n \geq 0\}$ is a DTMC. The state space is $S = \{0, 1, \cdots, M\}$. Now, for $0 \leq i < j \leq M$,

$$p_{i,j} = \mathsf{P}(\max\{X_n, Y_{n+1}\} = j | X_n = i) = \mathsf{P}(Y_n = j) = \alpha_j.$$

Also,

$$p_{i,i} = \mathsf{P}(\max\{X_n, Y_{n+1}\} = i | X_n = i) = \mathsf{P}(Y_n \leq i) = \sum_{k=0}^{i} \alpha_k.$$

2.18. Let $Y_n = u$ is the machine is up at time n and d if it is down at time n. If $Y_n = u$, let X_n be the remaining up time at time n; and if $Y_n = d$, let X_n be the remaining down time at time n. Then $\{(X_n, Y_n), n \geq 0\}$ is a DTMC with state space

$$S = \{(i, j) : i \geq 1, j = u, d\}$$

and transition probabilities

$$p_{(i,j),(i-1,j)} = 1, \ \ i \geq 2, j = u, d,$$

$$p_{(1,u),(i,d)} = d_i, \ p_{(1,d),(i,u)} = u_i, \ \ i \geq 1.$$

2.19. Let X_n be the number of messages in the inbox at 8:00am on day n. Ms. Friendly answers $Z_n = \text{Bin}(X_n, p)$ emails on day n. hence $X_n - Z_n = \text{Bin}(X_n, 1-p)$ emails are left for the next day. Y_n is the number messages that arrive during 24 hours on day n. Hence at the beginning of the next day there $X_{n+1} = Y_n + \text{Bin}(X_n, 1-p)$ in her mail box. Since $\{Y_n, n \geq 0\}$ is iid, $\{X_n, n \geq 0\}$ is a DTMC.

2.20. Let X_n be the number of bytes in this buffer in slot n, after the input during the slot and the removal (playing) of any bytes. We assume that the input during the slot occurs before the removal. Thus

$$X_{n+1} = \max\{\min\{X_n + A_{n+1}, B\} - 1, 0\}.$$

Thus if $X_n = 0$ and there is no input, $X_{n+1} = 0$. Similarly, if $X_n = B$, $X_{n+1} = B - 1$. The process $\{X_n, n \geq 0\}$ is a random walk on $\{0, ..., B-1\}$ with the following transition probabilities:

$$p_{0,0} = \alpha_0 + \alpha_1, \;\; p_{0,1} = \alpha_2,$$

$$p_{i,i-1} = \alpha_0, \;\; p_{i,i} = \alpha_1, \;\; p_{i,i+1} = \alpha_2, \;\; 0 < i < B-1,$$

$$p_{B-1,B-1} = \alpha_1 + \alpha_2; p_{B-1,B-2} = \alpha_0.$$

2.21. Let X_n be the number of passengers on the bus when it leaves the nth stop. Let D_{n+1} be the number of passengers that alight at the $(n+1)$st stop. Since each person on board the bus gets off with probability p in an independent fashion, D_{n+1} is $Bin(X_n, p)$ random variable. Also, $X_n - D_{n+1}$ is a $Bin(X_n, 1-p)$ random variable. Y_{n+1} is the number of people that get on the bus at the $(n+1)$st bus stop. Hence

$$X_{n+1} = \min\{X_n - D_{n+1} + Y_{n+1}, B\}.$$

Since $\{Y_n, n \geq 0\}$ is a sequence of iid random variables, it follows from the above recursive relationship, that $\{X_n, n \geq 0\}$ is a DTMC. The state space is $\{0, 1, ..., B\}$. For $0 \leq i \leq B$, and $0 \leq j < B$, we have

$$\begin{aligned}
p_{i,j} &= \mathsf{P}(X_{n+1} = j | X_n = i) \\
&= \mathsf{P}(X_n - D_{n+1} + Y_{n+1} = j | X_n = i) \\
&= \mathsf{P}(Y_{n+1} - Bin(i,p) = j - i) \\
&= \sum_{k=0}^{i} \mathsf{P}(Y_{n+1} - Bin(i,p) = j - i | Bin(i,p) = k) \mathsf{P}(Bin(i,p) = k) \\
&= \sum_{k=0}^{i} \mathsf{P}(Y_{n+1} = k + j - i | Bin(i,p) = k) \binom{i}{k} p^k (1-p)^{i-k} \\
&= \sum_{k=0}^{i} \binom{i}{k} p^k (1-p)^{i-k} \alpha_{k+j-i},
\end{aligned}$$

where we use the convention that $\alpha_k = 0$ if $k < 0$. Finally,

$$p_{i,B} = 1 - \sum_{j=0}^{B-1} p_{ij}.$$

2.22. The state space is $\{-1, 0, 1, 2, ..., k-1\}$. The system is in state -1 at time n if it is in economy mode after the n-th item is produced (and possibly inspected). It is in state i $(1 \leq i \leq k)$ if it is in 100% inspection mode and i consecutive non-defective items have been found so far. The transition probabilities are

$$p_{-1,0} = p/r, \;\; p_{-1,-1} = 1 - p/r,$$

$$p_{i,i+1} = 1 - p, \; p_{i,0} = p, \;\; 0 \leq i \leq k-2$$

$$p_{k-1,-1} = 1 - p, \; p_{k-1,0} = p.$$

2.23. X_n is the amount on hand at the beginning of the nth day, and D_n is the demand during the nth day. Hence the amount on hand at the end of the nth day is $X_n - D_n$. If this is s or more, no order is placed, and hence the amount on hand at the beginning of the $(n+1)$st day is $X_n - D_n$. On the other hand, if $X_n - D_n < s$, then the inventory is brought upto S at the beginning of the next day, thus making $X_{n+1} = S$. Thus

$$X_{n+1} = \begin{cases} X_n - D_n & \text{if } X_n - D_n \geq s, \\ S & \text{if } X_n - D_n < s. \end{cases}$$

Since $\{D_n, n \geq 0\}$ are iid, $\{X_n, n \geq 0\}$ is a DTMC on state space $\{s, s+1, ..., S-1, S\}$. We compute the transition probabilities next. For $s \leq j \leq i \leq S, j \neq S$, we have

$$\begin{aligned} \mathsf{P}(X_{n+1} = j | X_n = i) &= \mathsf{P}(X_n - D_n = j | X_n = i) \\ &= \mathsf{P}(D_n = i - j) = \alpha_{i-j}. \end{aligned}$$

and for $s \leq i < S, j = S$ we have

$$\begin{aligned} \mathsf{P}(X_{n+1} = S | X_n = i) &= \mathsf{P}(X_n - D_n < s | X_n = i) \\ &= \mathsf{P}(D_n > i - s) = \sum_{k=i-s}^{\infty} \alpha_k. \end{aligned}$$

Finally

$$\begin{aligned} \mathsf{P}(X_{n+1} = S | X_n = S) &= \mathsf{P}(X_n - D_n < s, \text{ or } X_n - D_n = S | X_n = S) \\ &= \mathsf{P}(D_n > S - s) + \mathsf{P}(D_n = 0) = \sum_{k=S-s+1}^{\infty} \alpha_k + \alpha_0. \end{aligned}$$

The transition probability matrix is given below:

$$P = \begin{bmatrix} \alpha_0 & 0 & 0 & \dots & 0 & b_0 & \\ \alpha_1 & \alpha_0 & 0 & \dots & 0 & b_1 & \\ \alpha_2 & \alpha_1 & \alpha_0 & \dots & 0 & b_2 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_{S-s-1} & \alpha_{S-s-2} & \alpha_{S-s-3} & \dots & \alpha_0 & b_{S-s-1} & \\ \alpha_{S-s} & \alpha_{S-s-1} & \alpha_{S-s-2} & \dots & \alpha_1 & \alpha_0 + b_S & \end{bmatrix},$$

where

$$b_j = \mathsf{P}(D_n > j) = \sum_{k=j+1}^{\infty} \alpha_k.$$

2.24. The state space of $\{(X_n, Y_n), n \geq 0\}$ is

$$S = \{(i,j) : i \geq 0, j = 1, 2\}.$$

Let

$$\beta_k^i = \sum_{j=k}^{\infty} \alpha_j^i, \quad k \geq 1, i = 1, 2.$$

The transition probabilities are given by (see solution to Modeling Exercise 2.1)

$$\begin{aligned}
p_{(i,1),(i+1,1)} &= \beta_{i+2}^1 / \beta_{i+1}^1, \quad i \geq 0, \\
p_{(i,2),(i+1,2)} &= \beta_{i+2}^2 / \beta_{i+1}^2, \quad i \geq 0, \\
p_{(i,1),(0,j)} &= v_j \alpha_{i+1}^1 / \beta_{i+1}^1, \quad i \geq 0, \\
p_{(i,2),(0,j)} &= v_j \alpha_{i+1}^2 / \beta_{i+1}^2, \quad i \geq 0.
\end{aligned}$$

2.25. X_n is the number of bugs in the program just before running it for the nth time. Suppose $X_n = k$. Then no is discovered on the nth run with probability $1-\beta_k$,a nd hence $X_{n+1} = k$. A bug will be discovered on the n run with probability β_k, in which case Y_n additional bugs are introduced, (with $\mathsf{P}(Y_n = i) = \alpha_i, \quad i = 0, 1, 2$) and $X_{n+1} = k - 1 + Y_n$. Hence, given $X_n = k$,

$$X_{n+1} = \begin{cases} k-1 & \text{with probability } \beta_k \alpha_0 = q_k \\ k & \text{with probability } \beta_k \alpha_1 + 1 - \beta_k = r_k \\ k+1 & \text{with probability } \beta_k \alpha_2 = p_k \end{cases}$$

Thus $\{X_n, n \geq 0\}$ is a DTMC with state space $\{0, 1, 2, ...\}$ with transition probability matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots \\ 0 & 0 & q_3 & r_3 & p_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

2.26. X_n = number of active rumor mongers at time n.
Y_n = number of individuals who have not heard the rumor up to and including time n.
Z_n = number of individuals who have heard the rumor up to and including time n, but have stopped spreading it.
The rumor spreading process is modeled as a three dimensional process $\{(X_n, Y_n, Z_n), n \geq 0\}$. We shall show that it is a DTMC.
Since the total number of individuals in the colony is N, we must have

$$X_n + Y_n + Z_n = N, \quad n \geq 0.$$

Now let A_n be the number of individuals who hear the rumor for the first time at time $n+1$. Now, an individual who has not heard the rumor by time n does not hear it by time $n+1$ if each the X_n rumor mongers at time n fails to contact him at time $n+1$. The probability of that is $((N-2)/(N-1))^{X_n}$. Hence

$$A_n \sim Bin(Y_n, 1 - ((N-2)/(N-1))^{X_n}).$$

Similarly, let B_n be the number of active rumor-mongers at time n that become inactive at tiem $n+1$. An active rumor monger becomes inactive if he contacts a person whos has already heard the rumor. The probability of that is $(X_n + Y_n - 1)/(N-1)$. Hence

$$B_n \sim Bin(X_n, (X_n + Y_n - 1)/(N-1)).$$

Now, from the definitions of the various random variables involved,

$$X_{n+1} = X_n - B_n + A_n,$$
$$Y_{n+1} = Y_n - A_n,$$
$$Z_{n+1} = Z_n + B_n.$$

Thus $\{(X_n, Y_n, Z_n), n \geq 0\}$ is a DTMC.

2.27. $\{X_n, n \geq 0\}$ is a DTMC with state space $S = \{rr, dr, dd\}$, since gene type of the $n+1$st generation only depends on that of the parents in the nth generation. We are given that $X_0 = rr$. Hence, the parents of the first generation are rr, dd. Hence X_1 is dr with probability 1. If X_n is dr, then the parents of the $(n+1)$st generation are dr and dd. Hence the $(n+1)$th generation is dr or dd with probability .5 each. Once the nth generation is dd it stays dd from then on. Hence transition probability matrix is given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & .5 & .5 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.28. Using the analysis in 2.27, we see that $\{X_n, n \geq 0\}$ is a DTMC with state space $S = \{rr, dr, dd\}$ with the following transition probability matrix:

$$P = \begin{bmatrix} .5 & .5 & 0 \\ .25 & .5 & .25 \\ 0 & .5 & .5 \end{bmatrix}.$$

2.29. Let X_n be the number of recipients in the nth generation. There are 20 recipients to begin with. Hence $X_0 = 20$. Let Y_i, n be the number of letters sent out by the ith recipient in the $n$the generation. The $\{Y_{i,n} : n \geq 0, i = 1, 2, ..., X_n\}$ are iid random variables with common pmf given below:

$$\mathsf{P}(Y_{i,n} = 0) = 1 - \alpha; \quad \mathsf{P}(Y_{i,n} = 20 = \alpha.$$

The number of recipients in the $(n+1)$st generation are given by

$$X_{n+1} = \sum_{i=1}^{X_n} Y_{i,n}.$$

Thus $\{X_n, n \geq 0\}$ is a branching process, following the terminology of Section 2.2.

Note that we cannot start with $X_0 = 1$ since we would need to use $Y_{1,0} = 20$ with probability 1, which is different distribution from the other $Y_{i,n}$s. This would invalidate the assumptions of a branching process.

2.30. Let X_n be the number of backlogged packets at the beginning of the nth slot. Furthermore, let I_n be the collision indicator defined as follows: $I_n = id$ if there are no transmissions in the $(n-1)$st slot (idle slot), $I_n = s$ if there is exactly 1 transmission in the $(n-1)$st slot (successful slot), and $I_n = e$ if there are 2 or more transmissions in the $(n-1)$st slot (error or collision in the slot). We shall model the state of the system at the beginning of the n the slot by (X_n, I_n). Now suppose $X_n = i, I_n = s$. Then, the backlogged packets retry with probability r. Hence, we get

$$\begin{aligned}
\mathsf{P}(X_{n+1} = i-1, I_{n+1} = s | X_n = i, I_n = s) &= (1-p)^{N-i} ir(1-r)^{i-1}, \\
\mathsf{P}(X_{n+1} = i, I_{n+1} = s | X_n = i, I_n = s) &= (N-i)p(1-p)^{N-i-1}(1-r)^i, \\
\mathsf{P}(X_{n+1} = i, I_{n+1} = id | X_n = i, I_n = s) &= (1-p)^{(N-i)}(1-r)^i, \\
\mathsf{P}(X_{n+1} = i, I_{n+1} = e | X_n = i, I_n = s) &= (1-p)^{(N-i)}(1-(1-r)^i - ir(1-r)^{i-1}). \\
\mathsf{P}(X_{n+1} = i+1, I_{n+1} = e | X_n = i, I_n = s) &= (N-i)p(1-p)^{N-i-1}(1-(1-r)^i) \\
\mathsf{P}(X_{n+1} = i+j, I_{n+1} = e | X_n = i, I_n = s) &= \binom{N-i}{j} p^i (1-p)^{N-i-j}, \ \ 2 \le j \le N-i.
\end{aligned}$$

Next suppose $X_n = i, I_n = id$. Then, the backlogged packets retry with probability $r'' > r$. The above equations become:

$$\begin{aligned}
\mathsf{P}(X_{n+1} = i-1, I_{n+1} = s | X_n = i, I_n = id) &= (1-p)^{N-i} ir'(1-r')^{i-1}, \\
\mathsf{P}(X_{n+1} = i, I_{n+1} = s | X_n = i, I_n = id) &= (N-i)p(1-p)^{N-i-1}(1-r')^i, \\
\mathsf{P}(X_{n+1} = i, I_{n+1} = id | X_n = i, I_n = id) &= (1-p)^{(N-i)}(1-r')^i, \\
\mathsf{P}(X_{n+1} = i, I_{n+1} = e | X_n = i, I_n = id) &= (1-p)^{(N-i)}(1-(1-r')^i - ir'(1-r')^{i-1}). \\
\mathsf{P}(X_{n+1} = i+1, I_{n+1} = e | X_n = i, I_n = id) &= (N-i)p(1-p)^{N-i-1}(1-(1-r')^i) \\
\mathsf{P}(X_{n+1} = i+j, I_{n+1} = e | X_n = i, I_n = id) &= \binom{N-i}{j} p^i (1-p)^{N-i-j}, \ \ 2 \le j \le N-i.
\end{aligned}$$

Finally, suppose $X_n = i, I_n = e$. Then, the backlogged packets retry with probability $r'' < r$. The above equations become:

$$\begin{aligned}
\mathsf{P}(X_{n+1} = i-1, I_{n+1} = s | X_n = i, I_n = e) &= (1-p)^{N-i} ir''(1-r'')^{i-1}, \\
\mathsf{P}(X_{n+1} = i, I_{n+1} = s | X_n = i, I_n = e) &= (N-i)p(1-p)^{N-i-1}(1-r'')^i, \\
\mathsf{P}(X_{n+1} = i, I_{n+1} = id | X_n = i, I_n = e) &= (1-p)^{(N-i)}(1-r'')^i, \\
\mathsf{P}(X_{n+1} = i, I_{n+1} = e | X_n = i, I_n = e) &= (1-p)^{(N-i)}(1-(1-r'')^i - ir''(1-r'')^{i-1}). \\
\mathsf{P}(X_{n+1} = i+1, I_{n+1} = e | X_n = i, I_n = e) &= (N-i)p(1-p)^{N-i-1}(1-(1-r'')^i) \\
\mathsf{P}(X_{n+1} = i+j, I_{n+1} = e | X_n = i, I_n = e) &= \binom{N-i}{j} p^i (1-p)^{N-i-j}, \ \ 2 \le j \le N-i.
\end{aligned}$$

This shows that $\{(X_n, I_n), n \ge 0\}$ is a DTMC with transition probabilities given above.

2.31. Let X_n be the number of packets ready for transmission at time n. Let Y_n be the number of packets that arrive during time $(n, n+1]$. If $X_n = 0$, no packets are transmitted during the nth slot and we have

$$X_{n+1} = Y_n.$$

If $X_n > 0$, exactly one packet is transmitted during the nth time slot. Hence,

$$X_{n+1} = X_n - 1 + Y_n.$$

Since $\{Y_n, n \geq 0\}$ are iid, we see that $\{X_n, n \geq 0\}$ is identical to the DTMC given in Example 2.16.

2.32. Let $Y_{i,n}$, $i = 1, 2$, be the number of non-defective items in the inventory of the ith machine at time n, after all production and any assembly at time n is done. Since the assembly is instantaneous, both $Y_{1,n}$ and $Y_{2,n}$ cannot be positive simultaneously. Now define

$$X_n = B_2 + Y_{1,n} - Y_{2,n}.$$

The state space of $\{X_n, n \geq 0\}$ is $S = \{0, 1, 2, ..., B_1 + B_2 - 1, M_1 + M_2\}$. Now,

$$X_n = k > B_2 \Rightarrow Y_{1,n} = k - B_2,\ Y_{2,n} = 0,$$

$$X_n = k < B_2 \Rightarrow Y_{1,n} = 0,\ Y_{2,n} = B_2 - k,$$

$$X_n = k = B_2 \Rightarrow Y_{1,n} = 0,\ Y_{2,n} = 0.$$

Thus X_n contains complete information about $Y_{1,n}$ and $Y_{2,n}$. $\{X_n, n \geq 0\}$ is a random walk on S as in Example 2.5 with

$$p_{n,n+1} = p_n = \begin{cases} \alpha_1 & \text{if } n = 0, \\ \alpha_1(1-\alpha_2) & \text{if } 0 < n < B_1 + B_2, \end{cases}$$

$$p_{n,n-1} = q_n = \begin{cases} \alpha_2 & \text{if } n = B_1 + B_2, \\ \alpha_2(1-\alpha_1) & \text{if } 0 < n < B_1 + B_2, \end{cases}$$

$$p_{n,n} = r_n = \begin{cases} 1-\alpha_1 & \text{if } n = 0, \\ \alpha_1\alpha_2 + (1-\alpha_1)(1-\alpha_2) & \text{if } 0 < n < B_1 + B_2, \\ 1-\alpha_2 & \text{if } n = B_1 + B_2. \end{cases}$$

2.33. Let X_n be the age of the light bulb in place at time n. Using the solution to Modeling Exercise 2.1, we see that $\{X_n, n \geq 0\}$ is a success-runs DTMC on $\{0, 1, ..., K-1\}$ with

$$q_i = p_{i+1}/b_{i+1}, p_i = 1 - q_i,\ \ 0 \leq i \leq K-2, q_{K-1} = 1,$$

where $b_i = P(Z_n \geq i) = \sum_{j=i}^{\infty} p_j$.

2.34. The same three models of reader behavior in Section 2.3.7 work if we consider a citation from paper i to paper j as link from webpage i to web page j, and action of visiting a page is taken to the same as actually looking up a paper.

Computational Exercises

2.1. Let X_n be the number of white balls in urn A after n experiments. $\{X_n, n \geq 0\}$ is a DTMC on $\{0, 1, ..., 10\}$ with the following transition probability matrix:

$$P = \begin{bmatrix} 0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.01 & 0.18 & 0.81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.04 & 0.32 & 0.64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.09 & 0.42 & 0.49 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.16 & 0.48 & 0.36 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0.50 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.36 & 0.48 & 0.16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.49 & 0.42 & 0.09 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.64 & 0.32 & 0.04 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.81 & 0.18 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 \end{bmatrix}.$$

Using the equation given in Example 2.21 we get the following table:

n	$X_0 = 8$	$X_0 = 5$	$X_0 = 3$
0	8.0000	5.0000	3.0000
1	7.4000	5.0000	3.4000
2	6.9200	5.0000	3.7200
3	6.5360	5.0000	3.9760
4	6.2288	5.0000	4.1808
5	5.9830	5.0000	4.3446
6	5.7864	5.0000	4.4757
7	5.6291	5.0000	4.5806
8	5.5033	5.0000	4.6645
9	5.4027	5.0000	4.7316
10	5.3221	5.0000	4.7853
11	5.2577	5.0000	4.8282
12	5.2062	5.0000	4.8626
13	5.1649	5.0000	4.8900
14	5.1319	5.0000	4.9120
15	5.1056	5.0000	4.9296
16	5.0844	5.0000	4.9437
17	5.0676	5.0000	4.9550
18	5.0540	5.0000	4.9640
19	5.0432	5.0000	4.9712
20	5.0346	5.0000	4.9769

2.2. Let P be the transition probability matrix and a the initial distribution given in the problem.

1. Let $a^{(2)}$ be the pmf of X_2. It is given by Equation 2.31. Substituting for a and P we get
$$a^{(2)} = [0.2050\ \ 0.0800\ \ 0.1300\ \ 0.3250\ \ 0.2600].$$

2.
$$\begin{aligned}\mathsf{P}(X_2 = 2, X_4 = 5) &= \mathsf{P}(X_4 = 5|X_2 = 2)\mathsf{P}(X_2 = 2)\\ &= \mathsf{P}(X_2 = 5|X_0 = 2) * (.0800)\\ &= [P^2]_{2,5} * (.0800)\\ &= (.0400) * (.0800) = .0032.\end{aligned}$$

3.
$$\begin{aligned}\mathsf{P}(X_7 = 3|X_3 = 4) &= \mathsf{P}(X_4 = 3|X_0 = 4)\\ &= [P^4]_{4,3}\\ &= .0318.\end{aligned}$$

4.
$$\begin{aligned}\mathsf{P}(X_1 \in \{1,2,3\}, X_2 \in \{4,5\}) &= \sum_{i=1}^{5} \mathsf{P}(X_1 \in \{1,2,3\}, X_2 \in \{4,5\}|X_0 = i)\mathsf{P}(X_0 = i)\\ &= \sum_{i=1}^{5} a_i \sum_{j=1}^{3}\sum_{k=4}^{5} \mathsf{P}(X_1 = j, X_2 = k\}|X_0 = i)\\ &= \sum_{i=1}^{5}\sum_{j=1}^{3}\sum_{k=4}^{5} a_i p_{i,j} p_{j,k}\\ &= .4450.\end{aligned}$$

2.3. Easiest way is to prove this by induction. Assume $a+b \neq 2$. Using the formula given in Computational Exercise 3, we see that

$$P^0 = \frac{1}{2-a-b}\begin{bmatrix} 1-b & 1-a \\ 1-b & 1-a \end{bmatrix} + \frac{1}{2-a-b}\begin{bmatrix} 1-a & a-1 \\ b-1 & 1-b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$P^1 = \frac{1}{2-a-b}\begin{bmatrix} 1-b & 1-a \\ 1-b & 1-a \end{bmatrix} + \frac{a+b-1}{2-a-b}\begin{bmatrix} 1-a & a-1 \\ b-1 & 1-b \end{bmatrix} = \begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix}.$$

Thus the formula is valid for $n = 0$ and $n = 1$. Now suppose it is valid for $n = k \geq 1$. Then

$$\begin{aligned}P^{k+1} &= P^k * P\\ &= \left[\frac{1}{2-a-b}\begin{bmatrix} 1-b & 1-a \\ 1-b & 1-a \end{bmatrix} + \frac{(a+b-1)^k}{2-a-b}\begin{bmatrix} 1-a & a-1 \\ b-1 & 1-b \end{bmatrix}\right] * \begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix}\\ &= \frac{1}{2-a-b}\begin{bmatrix} 1-b & 1-a \\ 1-b & 1-a \end{bmatrix} + \frac{(a+b-1)^{k+1}}{2-a-b}\begin{bmatrix} 1-a & a-1 \\ b-1 & 1-b \end{bmatrix},\end{aligned}$$

where the last equation follows after some algebra. Hence the formula is valid for

$n = k + 1$. Thus the result is established by induction.

If $a + b = 2$, we must have $a = b = 1$. Hence,

$$P = P^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The formula reduces to this after an application of L'Hopital's rule to compute the limit.

2.4. Let X_n be as defined in Example 2.1b. Then $\{X_n, n \geq 0\}$ is a DTMC with transition matrix given below:

$$P = \begin{bmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{bmatrix}.$$

Using the results of Computational exercise 3 above, we get

$$P^n = \frac{1}{2-p_1-p_2} \begin{bmatrix} 1-p_2 & 1-p_1 \\ 1-p_2 & 1-p_1 \end{bmatrix} + \frac{(p_1+p_2-1)^n}{2-p_1-p_2} \begin{bmatrix} 1-p_1 & p_1-1 \\ p_2-1 & 1-p_2 \end{bmatrix}.$$

Using the fact that the first patient is given a drug at random, we have

$$\mathsf{P}(X_1 = 1) = \mathsf{P}(X_1 = 2) = .5.$$

Hence, for $n \geq 1$, we have

$$\begin{aligned} \mathsf{P}(X_n = 1) &= \mathsf{P}(X_n = 1 | X_1 = 1) * .5 + \mathsf{P}(X_n = 1 | X_1 = 2) * .5 \\ &= \frac{1}{2} \cdot ([P^{n-1}]_{1,1} + [P^{n-1}]_{2,1}) \\ &= 1 - \frac{(p_1 - p_2) * ((p_1 + p_2 - 1)^{(n-1)} - 1)}{2 - a - b}. \end{aligned}$$

Now, let $Y_r = 1$ if the rth patient gets drug 1, and 0 otherwise. Then

$$Z_n = \sum_{r=1}^{n} Y_r$$

is the number of patients among the first n who receive drug 1. Hence

$$\begin{aligned} \mathsf{E}(Z_n) &= \mathsf{E}(\sum_{r=1}^{n} Y_r) \\ &= \sum_{r=1}^{n} \mathsf{E}(Y_r) \\ &= \sum_{r=1}^{n} \mathsf{P}(Y_r = 1) \\ &= \sum_{r=1}^{n} \mathsf{P}(X_r = 1) \end{aligned}$$

$$\begin{aligned} &= \sum_{r=1}^{n}\left[1+\frac{(p_2-p_1)*((p_1+p_2-1)^{(n-1)}-1)}{2-a-b}\right] \\ &= n\frac{2(1-p_2)}{2-p_1-p_2}-\frac{(p_1-p_2)}{(2-p_1-p_2)^2}\cdot((p_1+p_2-1)^n-1). \end{aligned}$$

2.5. Let X_n be the brand chosen by a typical customer in the nth week. Then $\{X_n, n \geq 0\}$ is a DTMC with transition probability matrix P given in Example 2.6. We are given the initial distribution a to be

$$a = [.3 \ \ .3 \ \ .4].$$

The distribution of X_3 is given by

$$a^{(3)} = aP^3 = [0.1317 \ \ 0.3187 \ \ 0.5496].$$

Thus a typical customer buys brand B in week 3 with probability .3187. Since all k customers behave independently of each other, the number of customers that buy brand B in week 3 is $B(k, .3187)$ random variable.

2.6. Since the machines are identical and independent, the total expected revenue over $\{0, 1, \cdots, n\}$ is given by $rM_{11}^{(n)}$, where $M^{(n)}$ is given in Example 2.24.

2.7. Let $\alpha = \frac{1+u}{1-d}$ and write

$$X_n = (1-d)^n \alpha^{Z_n}.$$

Using the results about the generating functions of a binomial, we get

$$\mathsf{E}(X_n) = (1-d)^n \mathsf{E}(\alpha^{Z_n}) = (1-d)^n (p\alpha + 1 - p)^n,$$

and

$$\mathsf{E}(X_n^2) = (1-d)^{2n} \mathsf{E}(\alpha^{2Z_n}) = (1-d)^{2n} (p\alpha^2 + 1 - p)^n.$$

This gives the mean and variance of X_n.

2.8. The initial distribution is

$$a = [1 \ \ 0 \ \ 0 \ \ 0].$$

(i) $a^{(2)} = aP^2 = [0.42 \ \ 0.14 \ \ 0.11 \ \ 0.33]$. Hence,

$$\mathsf{P}(X_2 = 4) = .33.$$

(ii) Since $\mathsf{P}(X_0 = 1) = 1$, we have

$$\begin{aligned} \mathsf{P}(X_1 = 2, X_2 = 4, X_3 = 1) &= \sum_{i=1}^{4} \mathsf{P}(X_1 = 2, X_2 = 4, X_3 = 1 | X_0 = i)\mathsf{P}(X_0 = i) \\ &= p(X_1 = 2, X_2 = 4, X_3 = 1 | X_0 = 1) \\ &= p_{1,2}p_{2,4}p_{4,1} \end{aligned}$$

$$= .015.$$

(iii) Using time homogeneity, we get

$$\begin{aligned} \mathsf{P}(X_7 = 4|X_5 = 2) &= \mathsf{P}(X_2 = 4|X_0 = 2) \\ &= [P^2]_{2,1} = .25 \end{aligned}$$

(iv) Let $b = [1234]'$. Then

$$\mathsf{E}(X_3) = a * P^3 * b = 2.455.$$

2.9. From the definition of X_n and Y_n we see that

$$X_{n+1} = \begin{cases} 20 & \text{if } X_n - Y_n < 10, \\ X_n - Y_n & \text{if } X_n - Y_n \geq 10. \end{cases}$$

Since $\{Y_n, n \geq 0\}$ are iid random variables, it follows that $\{X_n, n \geq 0\}$ is a DTMC on state space $\{10, 11, 12, ..., 20\}$. The transition probability matrix is given by

$$P = \begin{bmatrix} .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .9 \\ .2 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .7 \\ .3 & .2 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .4 \\ .4 & .3 & .2 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .4 & .3 & .2 & .1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .4 & .3 & .2 & .1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .4 & .3 & .2 & .1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .4 & .3 & .2 & .1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .4 & .3 & .2 & .1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .4 & .3 & .2 & .1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .4 & .3 & .2 & .1 \end{bmatrix}.$$

The initial distribution is

$$a = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1].$$

Let

$$b = [10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20]'.$$

Then we have

$$\mathsf{E}(X_n) = aP^n b, \quad n \geq 0.$$

Using this we get

n	$\mathsf{E}(X_n)$
0	20.0000
1	18.0000
2	16.0000
3	14.0000
4	13.1520
5	14.9942
6	16.5868
7	16.5694
8	15.4925
9	14.5312
10	14.5887

2.10. From Example 2.12, $\{X_n, n \geq 0\}$ is a random walk on $\{0, 1, 2, 3, ...\}$ with parameters

$$r_0 = 1 - p = .2, \quad p_0 = .8,$$
$$q_i = q(1-p) = .14, \quad p_i = p(1-q) = .24, \quad r_i = .62, \quad i \geq 1.$$

We are given $X_0 = 0$. Hence,

$$\mathsf{P}(X_1 = 0) = .2, \quad \mathsf{P}(X_1 = 1) = .8.$$

And,

$$\begin{aligned} \mathsf{P}(X_2 = 0) &= \mathsf{P}(X_2 = 0|X_1 = 0)\mathsf{P}(X_1 = 0) + \mathsf{P}(X_2 = 0|X_1 = 1)\mathsf{P}(X_1 = 1) \\ &= .2 * .2 + .14 * .8 \\ &= .152. \end{aligned}$$

2.11. The simple random walk of Example 2.19 has state space $\{0, \pm 1, \pm 2, ...\}$, and the following transition probabilities:

$$p_{i,i+1} = p, \quad p_{i,i-1} = q = 1 - p.$$

We want to compute

$$p_{i,j}^n = \mathsf{P}(X_n = j|X_0 = i).$$

Let R be the number of right steps taken by the random walk during the first n steps, and L be the number of right steps taken by the random walk during the first n steps. Then,

$$R + L = n, \quad R - L = j - i.$$

Thus

$$R = \frac{1}{2}(n + j - i), \quad L = \frac{1}{2}(n + i - j).$$

This is possible if and only if $n + j - i$ is even. There are $\binom{n}{R}$ ways of taking R steps to the right and L steps to the left in the first n steps. Hence, if $n+j-i$ is even, we have

$$p_{i,j}^n = \binom{n}{R} p^R q^L,$$

otherwise it is zero.

2.12. Let $\{X_n, n \geq 0\}$ be the DTMC of Modeling Exercise 2.5. Let $Y_n = (X_{n-1}, X_n)$. $\{Y_n, n \geq 1\}$ is a DTMC with transition probability matrix given below:

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ .75 & .25 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}.$$

Suppose the rainy spell starts on day 1, i.e $Y_1 = (1,2)$. Let R be the length of the rainy spell.Then

$$\mathsf{P}(R=1) = \mathsf{P}(Y_2 = (2,1)|Y_1 = (1,2)) = .5.$$

For $k \geq 2$ we have

$$\begin{aligned} \mathsf{P}(R=k) &= \mathsf{P}(Y_i = (2,2), i = 2,3,...,k, Y_{k+1} = (2,1)|Y_1 = (1,2)) \\ &= \mathsf{P}(Y_2 = (2,2)|Y_1 = (1,2)) \prod_{i=2}^{k-1} \mathsf{P}(Y_{i+1} = (2,2)|Y_i = (2,2))\mathsf{P}(Y_{k+1} = (2,1)|Y_k = (2,2)) \\ &= (.5)(.6)^{k-2}(.4). \end{aligned}$$

By a similar analysis, the distribution of the length of the sunny spell S, is given by

$$\mathsf{P}(S=1) = .25,$$

and, for $k \geq 2$,

$$\mathsf{P}(S=k) = (.75)(.8)^{k-2}(.2).$$

2.13. The matlab program to compute the quantities is given below.

```
*************************************************************
C = 20; % Capacity of the bus.
l = 10; % Passengers ata stop are P(l).
p = 0.4; % prob that a rider gets off at a stop.
N = 20; % Number of stops.
%pp(i+1) = p(a Poisson(l) rv = i).
pp= zeros(1,C+1);pp(1)=exp(-l);
for i = 1:C pp(i+1) = pp(i)*l/(i);
end;
% P = transition probability matrix of the DTMC {X_n, n ≥ 0}.
P = zeros(C+1);
```

```
for i = 0:C
%pb(j+1) = P(a Bin(i,p) rv = j).
pb=zeros(1,i+1);
pb(1) = p^(i);
for j=1:i
pb(j+1) = pb(j)*((1-p)/p)*(i-j+1)/(j);
end;
for j=0:C-1
P(i+1,j+1) = 0;
for k=0:min(i,j)
P(i+1,j+1) = P(i+1,j+1) + pb(k+1)*pp(j-k+1);
end;
end;
end;
b = sum(P');
P(:,C+1) = ones(C+1,1) - b';
% ex(n) = E(X_n).
nv=[];ex = [];b=[0:C]';a=[1 zeros(1,C)];
for n=0:N
nv = [nv n];
ex = [ex a*P^n*b];
end;
[nv' ex']
```
****

The final output is

n	$\mathsf{E}(X_n)$
0	0
1	9.9972
2	15.6322
3	17.9843
4	18.7442
5	18.9664
6	19.0294
7	19.0471
8	19.0520
9	19.0534
10	19.0538
11	19.0539
12	19.0539
13	19.0539
14	19.0539
15	19.0539
16	19.0539
17	19.0539
18	19.0539
19	19.0539
20	19.0539

2.14. Follows from direct verification that

$$Px_k = \lambda_k x_k, \;\; y_k P = \lambda_k y_k, \quad 1 \le k \le m.$$

2.15. The statement holds for $n = 1$. Now suppose it holds for a given $n \ge 1$. Then

$$p_{00}^{n+1} = \sum_{i=0}^{\infty} \mathsf{P}(X_n = i) p_{i0} = q \sum_{i=0}^{\infty} \mathsf{P}(X_n = i) = q.$$

The result is true by induction.

2.16. The matlab program is listed below:

```
*******************************************
N = 3; %Number of points on the circle.
p = .4; %probability of clockwise jump.
%P = the transition probability matrix.
P = zeros(N,N);
P(1,N) = 1-p;P(1,2) = p;
for i = 2:N-1
P(i,i+1) = p;
P(i,i-1) = 1-p;
```

```
end;
P(N,1) = p; P(N,N-1) = 1-p;
[V, D] = eig(P);
IV = inv(V);
for i=1:N
i
%i th eigenvalue is printed next.
D(i,i)
% the matrix B_i is printed next.
V(:,i)*IV(i,:)
end;
```

****

The output of the above program is

$$\lambda_1 = 1, \quad \lambda_2 = -.5 + .1732i, \quad \lambda_3 = -.5 - .1732i.$$

The corresponding matrices are

$$B_1 = \begin{bmatrix} 0.3333 & 0.3333 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.3333 & -0.1667 + 0.2887i & -0.1667 - 0.2887i \\ -0.1667 - 0.2887i & 0.3333 & -0.1667 + 0.2887i \\ -0.1667 + 0.2887i & -0.1667 - 0.2887i & 0.3333 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0.3333 & -0.1667 - 0.2887i & -0.1667 + 0.2887i \\ -0.1667 + 0.2887i & 0.3333 & -0.1667 - 0.2887i \\ -0.1667 - 0.2887i & -0.1667 + 0.2887i & 0.3333 \end{bmatrix}.$$

Then

$$P^n = B_1 + (-.5 + .1732i)^n B_2 + (-.5 - .1732i)^n B_3.$$

2.18. We show by induction that

$$p_{ij}^{(n)} = qp^j, \quad j = 0, 1, 2, ..., n-1,$$

$$p_{i,i+n} = p^n.$$

All other $p_{i,j}^{(n)}$ are zero. This is clearly true at n=1. We show that if it holds for n, it holds for $n + 1$. For $j = 0, 1, ..., n - 1$

$$\begin{aligned} p_{i,j}^{(n+1)} &= qp_{0,j}^{(n)} + pp_{i+1,j}^{(n)} \\ &= qqp^j + pqp^j = qp^j. \end{aligned}$$

For $j = n$, we hace

$$\begin{aligned} p_{i,n}^{(n+1)} &= qp_{0,n}^{(n)} + pp_{i+1,n}^{(n)} \\ &= qp^n + 0 = qp^n. \end{aligned}$$

Finally,

$$\begin{aligned} p_{i,i+n+1}^{(n+1)} &= qp_{0,n+1}^{(n)} + pp_{i+1,i+1+n}^{(n)} \\ &= 0 + pp^n = p^{n+1}. \end{aligned}$$

Thus the result holds for $n + 1$. Hence it holds for all n by induction.

2.19. We are given

$$P = \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}.$$

Hence

$$I - zP = \begin{bmatrix} 1-0.3z & -0.4z & -0.3z \\ -0.4z & 1-0.5z & -0.1z \\ -0.6z & -0.2z & 1-0.2z \end{bmatrix}.$$

Following Example 2.15, we get

$$\begin{aligned} \sum_{n=0}^{\infty} p_{11}^{(n)} z^n &= (I - zP)_{11}^{-1} \\ &= \frac{det(A)}{det(I - zP)} \end{aligned}$$

where

$$A = \begin{bmatrix} 1-0.5z & -0.1z \\ -0.2z & 1-0.2z \end{bmatrix}.$$

Expanding, we get

$$\begin{aligned} \sum_{n=0}^{\infty} p_{11}^{(n)} z^n &= \frac{1 - 0.7z + .08z^2}{(1 - z - .05z^2 + .05z^3} \\ &= \frac{.4}{1-z} + \frac{0.5236}{1+0.2236z} - \frac{0.0764}{1-.2236z} \\ &= \sum_{n=0}^{\infty} (.4 + 0.5236(-0.2236)^n + 0.0764(0.2236)^n) z^n \end{aligned}$$

Hence, we get

$$p_{11}^{(n)} = .4 + 0.5236(-0.2236)^n + 0.0764(0.2236)^n, \quad n \geq 0.$$

2.20. From the structure of the DTMC

$$p_{ij}^{(n)} = 0 \text{ if } j > i.$$

Also,

$$p_{ii}^{(n)} = \left(\frac{1}{i+1}\right)^n, \quad i \geq 0.$$

Hence,

$$\begin{aligned}\phi_{ii}(z) &= \sum_{n=0}^{\infty} p_{ii}^{(n)} z^n \\ &= \sum_{n=0}^{\infty} \left(\frac{z}{i+1}\right)^n \\ &= 1/(1 - z/(i+1)).\end{aligned}$$

Next,

$$\begin{aligned}\phi_{i,i-1}(z) &= \sum_{n=0}^{\infty} p_{i,i-1}^{(n)} z^n \\ &= \sum_{n=0}^{\infty} \{\frac{1}{i+1} p_{i,i-1}^{(n-1)} + \frac{1}{i+1} p_{i-1,i-1}^{(n-1)}\} z^n \\ &= \frac{z}{i+1}\phi_{i,i-1}(z) + \frac{z}{i+1}\phi_{i-1,i-1}(z).\end{aligned}$$

Solving the above we get

$$\phi_{i,i-1}(z) = \frac{z}{i+1}\left\{(1 - \frac{z}{i+1})(1 - \frac{z}{i})\right\}^{-1}.$$

In general we have

$$\phi_{i,j} = \frac{z}{i+1}(\sum_{k=j}^{i} \phi_{k,j}(z)).$$

The result follows from this by induction.

2.21. The $\{X_n, n \geq 0\}$ as defined in Modeling Exercise 2.28 is a 3-state DTMC with transition probability matrix given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & .5 & .5 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using Matlab, we get

$$P = XDX^{-1},$$

where

$$X = \begin{bmatrix} 1.0000 & 0.8944 & 1 \\ 0 & 0.4472 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$X^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, $n \geq 1$,

$$P^n = XD^nX^{-1} = \begin{bmatrix} 0 & 2^{1-n} & 1-2^{1-n} \\ 0 & 2^{-n} & 1-2^{-n} \\ 0 & 0 & 1 \end{bmatrix}.$$

2.22. The $\{X_n, n \geq 0\}$ as defined in Modeling Exercise 2.27 is a 3-state DTMC with transition probability matrix given by

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.25 & .5 & .25 \\ 0 & 0.5 & 0.5 \end{bmatrix}.$$

Using Matlab, we get

$$P = XDX^{-1},$$

where

$$X = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$

and

$$X^{-1} = \begin{bmatrix} 0.25 & -0.50 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.00 & 0 & 0.50 \end{bmatrix}.$$

Hence, $n \geq 1$,

$$P^n = XD^nX^{-1} = \begin{bmatrix} .25+2^{-n-1} & .50 & .25-2^{-1-n} \\ 0.25 & 0.50 & 0.25 \\ .25-2^{-1-n} & 0.50 & .25+2^{-n-1} \end{bmatrix}.$$

2.23. Let $\{X_n, n \geq 0\}$ be a branching process with $X_0 = i$. Suppose the individuals in the zeroth generation are indexed $1, 2, ..., i$. Let X_n^k be the number of individuals in the nth generation that are direct descendants of the kth individual in generation zero. Then,

$$X_0^k = 1, \quad 1 \leq k \leq i,$$

and

$$X_n = \sum_{k=1}^{i} X_n^k, \quad n \geq i.$$

Since the offsprings do not interact with each other, it is clear that $\{X_n^k, n \geq 0\}$,

$1 \leq k \leq i$ are i independent and stochastically identical branching processes, each beginning with a single individual. Hence,

$$\begin{aligned} \mathsf{E}(X_n) &= \mathsf{E}(\sum_{k=1}^{i} X_n^k) \\ &= \sum_{k=1}^{i} \mathsf{E}(X_n^k) \\ &= i\mu^n, \end{aligned}$$

from Equation 2.37. Similarly

$$\begin{aligned} Var(X_n) &= Var(\sum_{k=1}^{i} X_n^k) \\ &= \sum_{k=1}^{i} Var(X_n^k) \\ &= \begin{cases} in\sigma^2 & \text{if } \mu = 1, \\ i\mu^{n-1}\sigma^2 \frac{\mu^n - 1}{\mu - 1} & \text{if } \mu \neq 1. \end{cases} \end{aligned}$$

2.24. The transition probability matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finding the eigenvalues and eigenvectors, we get, with $\theta = \sqrt{pq}$,

$$D = \begin{bmatrix} \theta & 0 & 0 & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$X = \begin{bmatrix} 0 & 0 & 1 - pq & -p^2 \\ \theta & -\theta & q & 0 \\ q & q & q^2 & pq \\ 0 & 0 & 0 & q \end{bmatrix}.$$

Then we get

$$P^n = XD^nX^{-1}.$$

2.25. The Wright-Fisher model satisfies

$$X_{n+1} \sim \text{Bin}(N, X_n/N).$$

Hence

$$\mathsf{E}(X_{n+1}) = N\mathsf{E}(X_n/N) = \mathsf{E}(X_n).$$

Hence $\mathsf{E}(X_n) = \mathsf{E}(X_0) = i$ for all $n \geq 0$. We next have

$$\mathsf{E}(X_{n+1}^2|X_n) = N\frac{X_n}{N}(1 - \frac{X_n}{N}).$$

Taking expectations again, we get

$$\mathsf{E}(X_{n+1}^2) = \mathsf{E}(X_n) + \mathsf{E}(X_n^2)(1 - \frac{1}{N}).$$

Using $\mathsf{E}(X_n) = i$, $a = 1 - \frac{1}{N}$ and solving recursively, we get

$$\mathsf{E}(X_n^2) = Ni(1 - a^n) + i^2 a^n, \quad n \geq 0.$$

Hence

$$\mathrm{Var}(X_n) = \mathsf{E}(X_n^2) - (\mathsf{E}(X_n))^2 = (1 - a^n)(Ni - i^2).$$

2.26. We have $X_{n+1} = X_n + 1$ with probability $p(X_n) = X_n(N - X_n)/N^2$, $X_{n+1} = X_n - 1$ with probability $p(X_n)$, and $X_{n+1} = X_n$ with the remaining probability. Hence

$$\mathsf{E}(X_{n+1}) = \mathsf{E}(X_n) = \mathsf{E}(X_0) = i.$$

Also,

$$\mathsf{E}(X_{n+1}^2) = \mathsf{E}((X_n^2 + 2X_n + 1)p(X_n) + X_n^2(1 - 2p(X_n)) + (X_n^2 - 2X_n + 1)p(X_n)).$$

Using $\mathsf{E}(X_n) = i$ and simplifying the above, we get

$$\mathsf{E}(X_{n+1}^2) = \mathsf{E}(X_n^2)(1 - \frac{2}{N^2}) + \frac{2i}{N}.$$

Using $\mathsf{E}(X_0^2) = i^2$, the above equation can be solved recursively to get

$$\mathsf{E}(X_n^2) = a^n i^2 + b\frac{1 - a^n}{1 - a}, \quad n \geq 0,$$

where $a = 1 - 2/N^2$, and $b = 1i/N$.

Conceptual Exercises

2.1. We have

$$\begin{aligned}
&\mathsf{P}(X_{n+2} = k, X_{n+1} = j | X_n = i, X_{n-1}, \cdots, X_0) \\
&= \mathsf{P}(X_{n+2} = k | X_{n+1} = j, X_n = i, X_{n-1}, \cdots, X_0) \cdot \mathsf{P}(X_{n+1} = j | X_n = i, X_{n-1}, \cdots, X_0) \\
&= \mathsf{P}(X_{n+2} = k | X_{n+1} = j)\mathsf{P}(X_{n+1} = j | X_n = i) \\
&= p_{jk}p_{ij} = \mathsf{P}(X_2 = k, X_1 = j | X_0 = i)
\end{aligned}$$

The result follows by summing over $j \in A$ and $k \in B$.

2.2. (a). Let $\{X_n, n \geq 0\}$ and $\{Y_n, n \geq 0\}$ be two independent DTMCs on state space $\{0, 1\}$ with transition probability matrices $P1$ and $P2$, where

$$P1 = \begin{bmatrix} 0.8 & 0.2 \\ .5 & .5 \end{bmatrix},$$

$$P2 = \begin{bmatrix} 0.3 & 0.7 \\ .4 & .6 \end{bmatrix}.$$

Both DTMCs start with initial distribution $[.5 .5]$. Let $Z_n = X_n + Y_n$. Now

$$\mathsf{P}(Z_2 = 2 | Z_1 = 1, Z_0 = 0) = \frac{\mathsf{P}(Z_2 = 2, Z_1 = 1, Z_0 = 0)}{\mathsf{P}(Z_1 = 1, Z_0 = 0)}.$$

We have

$$\begin{aligned}
&\mathsf{P}(Z_2 = 2, Z_1 = 1, Z_0 = 0) \\
&= \mathsf{P}(X_2 = Y_2 = 1, X_1 + Y_1 = 1, X_0 = Y_0 = 0) \\
&= \mathsf{P}(X_2 = Y_2 = 1, X_1 = 1, Y_1 = 0, X_0 = Y_0 = 0) + \mathsf{P}(X_2 = Y_2 = 1, X_1 = 0, Y_1 = 1, X_0 = Y_0 = 0) \\
&= \mathsf{P}(X_2 = 1, X_1 = 1, X_0 = 0)\mathsf{P}(Y_2 = 1, Y_1 = 0, Y_0 = 0) \\
&\quad + \mathsf{P}(X_2 = 1, X_1 = 0, X_0 = 0)p(Y_2 = 1, Y_1 = 1, Y_0 = 0) \\
&= (.5)(.1)(.5)(.21) + (.5)(.16)(.5)(.42) = .0221.
\end{aligned}$$

Similarly

$$\mathsf{P}(Z_1 = 1, Z_0 = 0) = .1550.$$

Hence

$$\mathsf{P}(Z_2 = 2 | Z_1 = 1, Z_0 = 0) = .0221/.155 = .1423.$$

However,

$$\mathsf{P}(Z_2 = 2, Z_1 = 1) = .0690,$$
$$\mathsf{P}(Z_1 = 1) = .5450.$$

Hence

$$p(Z_2 = 2 | Z_1 = 1) = .0690/.5450 = .1266.$$

Thus $\{Z_n, n \geq 0\}$ is not a DTMC.

(b). Let $\{X_n, n \geq 0\}$ and $\{Y_n, n \geq 0\}$ be two independent DTMCs with state space S^1 and S^2 and transition probability matrices P^1 and P^2, respectively. Let $Z_n = (X_n, Y_n)$. The state space of $\{Z_n, n \geq 0\}$ is $S^1 \times S^2$. Furthermore,

$$\begin{aligned}
& \mathsf{P}(Z_{n+1} = (j,l)|Z_n = (i,k), Z_{n-1}, ..., Z_0) \\
= \; & \mathsf{P}(X_{n+1} = j, Y_{n+1} = l|X_n = i, Y_n = k, X_{n-1}, Y_{n-1}, ..., X_0, Y_0) \\
= \; & \mathsf{P}(X_{n+1} = j|X_n = i, X_{n-1}, ..., X_0) \cdot \mathsf{P}(Y_{n+1} = l|Y_n = k, Y_{n-1}, ..., Y_0) \\
= \; & \mathsf{P}(X_{n+1} = j|X_n = i) \cdot \mathsf{P}(Y_{n+1} = l|Y_n = k) \\
= \; & P^1_{i,j} \cdot P^2_{k,l} \\
= \; & \mathsf{P}(Z_{n+1} = (j,l)|Z_n = (i,k)).
\end{aligned}$$

Thus $\{Z_n, n \geq 0\}$ is a DTMC.

2.3. (a). False. Let $\{X_n, n \geq 0\}$ be a DTMC with state space $\{1, 2, 3\}$ and transition probability matrix

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & .5 & .5 \\ .75 & .25 & 0 \end{bmatrix}.$$

Let the initial distribution be $a = [.2 0 .8]$. Now

$$\begin{aligned}
\mathsf{P}(X_2 = 1|X_1 \in \{1,2\}, X_0 = 1) &= \frac{\mathsf{P}(X_2 = 1, X_1 \in \{1,2\}, X_0 = 1)}{\mathsf{P}(X_1 \in \{1,2\}, X_0 = 1)} \\
&= \frac{.1280}{.2} = .64.
\end{aligned}$$

However,

$$\begin{aligned}
\mathsf{P}(X_2 = 1|X_1 \in \{1,2\}) &= \frac{\mathsf{P}(X_2 = 1, X_1 \in \{1,2\})}{\mathsf{P}(X_1 \in \{1,2\})} \\
&= \frac{.4800}{1} = .4800.
\end{aligned}$$

(b). True. We have

$$\begin{aligned}
& \mathsf{P}(X_n = j_0|X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k) \\
= \; & \frac{\mathsf{P}(X_n = j_0, X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)}{\mathsf{P}(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)} \\
= \; & \frac{\mathsf{P}(X_{n+2} = j_2, ..., X_{n+k} = j_k|X_n = j_0, X_{n+1} = j_1)\mathsf{P}(X_n = j_0, X_{n+1} = j_1)}{\mathsf{P}(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)} \\
= \; & \frac{\mathsf{P}(X_{n+2} = j_2, ..., X_{n+k} = j_k|X_{n+1} = j_1)\mathsf{P}(X_n = j_0, X_{n+1} = j_1)}{\mathsf{P}(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)} \\
= \; & \frac{\mathsf{P}(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)\mathsf{P}(X_n = j_0, X_{n+1} = j_1)/\mathsf{P}(X_{n+1} = j_1)}{\mathsf{P}(X_{n+1} = j_1, X_{n+2} = j_2, ..., X_{n+k} = j_k)} \\
= \; & \mathsf{P}(X_n = j_0, X_{n+1} = j_1)/\mathsf{P}(X_{n+1} = j_1)
\end{aligned}$$

$$= \quad \mathsf{P}(X_n = j_0 | X_{n+1} = j_1).$$

(c). False. Time shifting is allowed only in conditional probabilities, not in joint probabilities. Consider the special case of $k = 0$. Then the equation reduces to

$$\mathsf{P}(X_n = j_0) = \mathsf{P}(X_0 = j_0).$$

This is clearly not valid in general. For the DTMC in part (a), for example, $\mathsf{P}(X_0 = 1) = .2$, but $\mathsf{P}(X_1 = 1) = .76$

2.4 (a). False. (True only if k = 0). b and the transition probability matrix will completely describe $\{X_n, n \geq k\}$. It does not determine distribution of X_{k-1} for example.

(b). False. (True only if f is one-to-one function, in which case $f(X_n)$ is a relabeled version of X_n.) As a counterexample, consider the DTMC in part (a). Let

$$f(1) = f(2) = 1, f(3) = 2.$$

Then $Y_n = 1$ if $X_n \in \{1, 2\}$, and $Y_n = 2$ if $X_n = 3$. The numerical calculations in part (a) show that $\{Y_n, n \geq 0\}$ is not a DTMC.

2.5. $\{(X_n, Y_n, Z_n), n \geq 0\}$ is a DTMC. Let

$$f(i, k, 0) = i, \quad f(i, k, 1) = k.$$

Then

$$W_n = f(X_n, Y_n, Z_n).$$

Thus $\{W_n, n \geq 0\}$ will be a DTMC if and only if distribution of $(X_{n+1}, Y_{n+1}, Z_{n+1})$ given $(X_n = i, Y_n = k, Z_n = 1)$ depends only on i, and that of $(X_{n+1}, Y_{n+1}, Z_{n+1})$ given $(X_n = i, Y_n = k, Z_n = 2)$ depends only on k. This won't be the case in general. Hence $\{W_n, n \geq 0\}$ is not a DTMC.

2.6. Let

$$\alpha_k = \mathsf{P}(Y_n = k), \quad k \in \{0, 1, 2, ...\}.$$

X_n be the value of the nth record. We have

$$\mathsf{P}(X_{n+1} = j | X_n = i, X_{n-1}, ..., X_0) = \frac{\alpha_j}{1 - f_i}, \quad j > i,$$

where

$$f_i = \mathsf{P}(Y_n \leq i).$$

Hence $\{X_n, n \geq 0\}$ is a DTMC.

2.7. Let

$$N_i = \min\{n \geq 0 : X_n \neq i\}.$$

Then, for $r \geq 1$,

$$\begin{aligned}\mathsf{P}(N_i = r | X_0 = i) &= \mathsf{P}(X_1 = i, ..., X_{r-1} = i, X_r \neq i | X_0 = i) \\ &= (p_{i,i})^{r-1}(1 - p_{i,i}).\end{aligned}$$

Thus the sojourn time in state i is $G(1 - p_{i,i})$.

2.8. By its definition, the rth visit of the DTMC $\{X_n, n \geq 0\}$ to the set A takes place at time N_r. (The zeroth visit is at time 0.) The actual state visited at this rth visit is Y_r. Thus the state space of $\{Y_r, r \geq 0\}$ is A. Then

$$\begin{aligned}\mathsf{P}(Y_{r+1} = j | Y_r = i, Y_{r-1}, ..., Y_0) &= \mathsf{P}(X_{N_{r+1}} = j | X_{N_r} = i, X_{N_{r-1}}, ..., X_{N_0}) \\ &= \mathsf{P}(X_{N_{r+1}} = j | X_{N_r} = i).\end{aligned}$$

Hence $\{Y_r, r \geq 0\}$ is a DTMC.

2.9. Let $a_i = \mathsf{P}(X_0 = i)$. Then

$$\mathsf{P}(X_1 = j) = \sum_{i \in S} \mathsf{P}(X_1 = j | X_0 = i) a_i = p \sum_{i \in S} a_i = p.$$

Suppose $\mathsf{P}(X_k = j) = p$ for some $k \geq 1$.

$$\mathsf{P}(X_{k+1} = j) = \sum_{i \in S} \mathsf{P}(X_{k+1} = j | X_k = i) \mathsf{P}(X_k = i) = p \sum_{i \in S} \mathsf{P}(X_k = i) = p.$$

Thus the result follows by induction.

2.10. Define

$$Y_n = \begin{cases} (X_n, 1) & \text{if } n \text{ is odd} \\ (X_n, 0) & \text{if } n \text{ is even.} \end{cases}$$

Then

$$\mathsf{P}(Y_{n+1} = (j,1) | Y_n = (i,0), Y_{n-1}, ..., Y_0) = \mathsf{P}(X_{n+1} = j | X_n = i, n \text{ even}) = a_{i,j},$$

and

$$\mathsf{P}(Y_{n+1} = (j,0) | Y_n = (i,1), Y_{n-1}, ..., Y_0) = \mathsf{P}(X_{n+1} = j | X_n = i, n \text{ odd}) = b_{i,j}.$$

Thus $\{Y_n, n \geq 0\}$ is a DTMC with state space $S \times \{0, 1\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

2.11. The solution of this problem is from Ross (Stochastic Processes, Wiley, 1983), Chapter 4, Section 1. Suppose $X_0 = 0$. We shall first show that

$$\mathsf{P}(X_n = i | |X_n| = i, |X_{n-1}|, ..., |X_0|) = \frac{p^i}{p^i + q^i}.$$

To prove this let

$$T = \max\{k : 0 \leq k \leq n, X_k = 0\}.$$

Then, since $X_T = 0$, we have

$$\mathsf{P}(X_n = i||X_n| = i, |X_{n-1}|, ..., |X_0|) = \mathsf{P}(X_n = i||X_n| = i, |X_{n-1}|, ..., |X_{T+1}|, X_T = 0).$$

From the definition of T, it follows that the event $E = \{|X_n| = i, |X_{n-1}| = i_{n-1}, ..., |X_{T+1}| = i_{T+1}, X_T = 0\}$ is teh union of two disjoint events $E_+ = \{X_n = i, X_{n-1} = i_{n-1}, ..., X_{T+1} = i_{T+1}, X_T = 0\}$, and $E_- = \{X_n = -i, X_{n-1} = -i_{n-1}, ..., X_{T+1} = -i_{T+1}, X_T = 0\}$. We have

$$\mathsf{P}(E_+) = p^{(n-T+i)/2} q^{(n-T-i)/2},$$
$$\mathsf{P}(E_-) = p^{(n-T-i)/2} q^{(n-T+i)/2}.$$

Hence

$$\mathsf{P}(X_n = i|E) = \frac{\mathsf{P}(E_+)}{\mathsf{P}(E_+) + \mathsf{P}(E_-)} = \frac{p^i}{p^i + q^i}.$$

Thus

$$\begin{aligned}
\mathsf{P}(|X_{n+1}| = i+1 \quad &| \quad |X_n| = i, |X_{n-1}|, ..., |X_0|) \\
&= \quad \mathsf{P}(X_{n+1} = i+1|X_n = i)\frac{p^i}{p^i + q^i} \\
&+ \quad \mathsf{P}(X_{n+1} = -(i+1)|X_n = -i)\frac{q^i}{p^i + q^i} \\
&= \quad \frac{p^{i+1} + q^{i+1}}{p^i + q^i}.
\end{aligned}$$

Thus $\{|X_n|, n \geq 0\}$ is a random walk on $\{0, 1, 2, ...\}$ with

$$p_{0,1} = 1,$$

and, for $i \geq 1$,

$$p_{i,i+1} = \frac{p^{i+1} + q^{i+1}}{p^i + q^i} = 1 - p_{i,i-1}.$$

2.12. A given partition $\{A_r\}$ of S is called lumpable if, for all A_r and A_s in the partition,

$$\sum_{j \in A_s} p_{i,j} = \alpha_{r,s}, \text{ for all } i \in A_r.$$

Now define

$$A_i = \{j \in S : f(j) = i\}.$$

$\{Y_n = f(X_n), n \geq 0\}$ is a DTMC if the partition $\{A_r\}$ is lumpable. To prove sufficiency, suppose $\{A_r\}$ is lumpable. Then

$$\begin{aligned}
\mathsf{P}(Y_{n+1} = s|Y_n = r, Y_{n-1}, ..., Y_0) &= \mathsf{P}(X_{n+1} \in A_s|X_n \in A_r, Y_{n-1}, ..., Y_0) \\
&= \alpha_{r,s}.
\end{aligned}$$

Necessity follows in a similar fashion.