2000

Solutions Manual: Chapter 2

7th Edition

Feedback Control of Dynamic Systems

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Chapter 2

Dynamic Models

Problems and Solutions for Section 2.1

1. Write the differential equations for the mechanical systems shown in Fig. 2.41. For (a) and (b), state whether you think the system will eventually decay so that it has no motion at all, given that there are non-zero initial conditions for both masses, and give a reason for your answer.

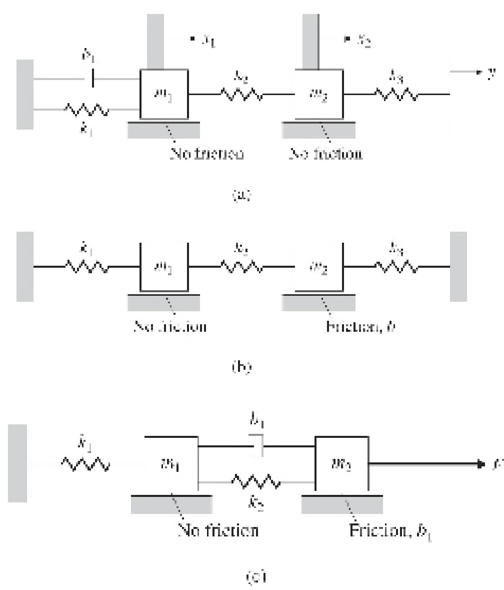
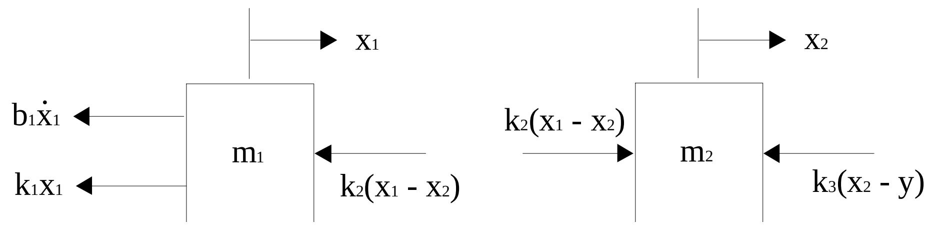


Fig. 2.41 Mechanical systems

Solution:

The key is to draw the Free Body Diagram (FBD) in order to keep the signs right. For (a), to identify the direction of the spring forces on the object, let $x_2 = 0$ and fixed and increase x_1 from 0. Then the k_1 spring will be stretched producing its spring force to the left and the k_2 spring will be compressed producing its spring force to the left also. You can use the same technique on the damper forces and the other mass.

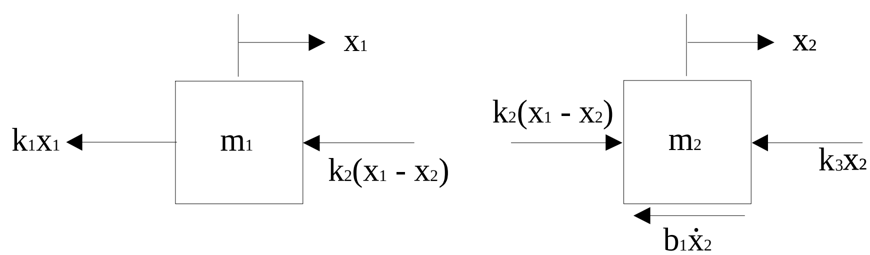


Free body diagram for Problem 2.1(a)

(a)

$$
\begin{aligned}
m_1 \ddot{x}_1 &= -k_1 x_1 - b_1 \dot{x}_1 - k_2 (x_1 - x_2) \\
m_2 \ddot{x}_2 &= -k_2 (x_2 - x_1) - k_3 (x_2 - y)
\end{aligned}
$$

There is friction affecting the motion of mass 1 which will continue to take energy out of the system as long as there is any movement of x_1.Mass 2 is undamped; therefore it will tend to continue oscillating. However, its motion will drive mass 1 through the spring; therefore, the entire system will continue to lose energy and will eventually decay to zero motion for both masses.



Free body diagram for Problem 2.1(b)

$$
\begin{aligned}
m_1 \ddot{x}_1 &= -k_1 x_1 - k_2 (x_1 - x_2) \\
m_2 \ddot{x}_2 &= -k_2 (x_2 - x_1) - b_1 \dot{x}_2 - k_3 x_2
\end{aligned}
$$

Again, there is friction on mass 2 so there will continue to be a loss of energy as long as there is any motion; hence the motion of both masses will eventually decay to zero.

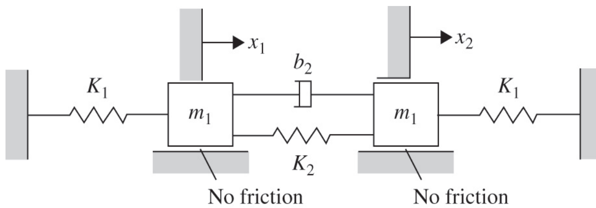
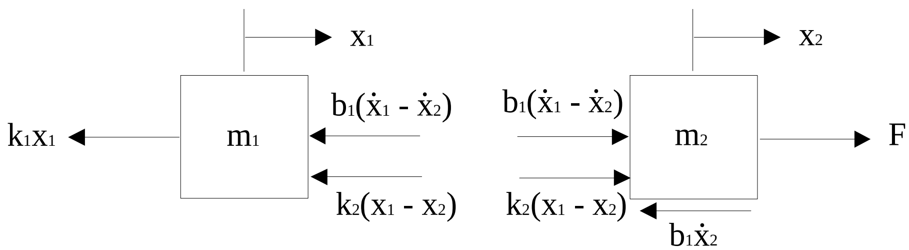


Figure 2.42: Mechanical system for Problem 2.2



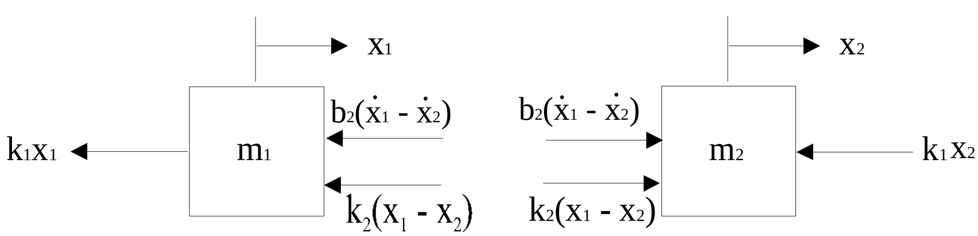
Free body diagram for Problem 2.1 (c)

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) \\
m_2\ddot{x}_2 &= F - k_2(x_2 - x_1) - b_1(\dot{x}_2 - \dot{x}_1) - b_1\dot{x}_2
\end{aligned}
$$

2. Write the differential equations for the mechanical systems shown in Fig. 2.42. State whether you think the system will eventually decay so that it has no motion at all, given that there are non-zero initial conditions for both masses, and give a reason for your answer.

   Solution:

   The key is to draw the Free Body Diagram (FBD) in order to keep the signs right. To identify the direction of the spring forces on the left side object, let $x_2 = 0$ and increase x_1 from 0. Then the k_1 spring on the left will be stretched producing its spring force to the left and the k_2 spring will be compressed producing its spring force to the left also. You can use the same technique on the damper forces and the other mass.



Free body diagram for Problem 2.2

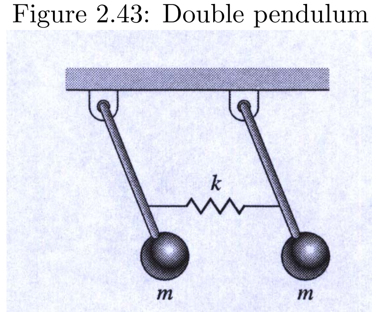
Then the forces are summed on each mass, resulting in

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) \\ m_2\ddot{x}_2 &= k_2(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) - k_1x_2 \end{aligned}$$

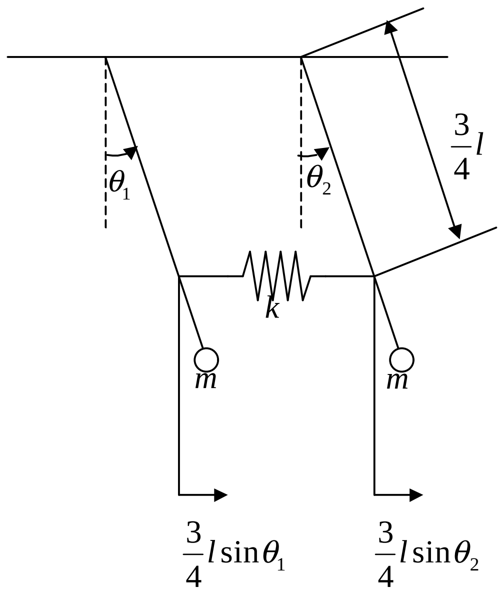
The *relative* motion between x_1 and x_2 will decay to zero due to the damper. However, the two masses will continue oscillating together without decay since there is no friction opposing that motion and flexure of the end springs is all that is required to maintain the oscillation of the two masses. However, note that the two end springs have the same spring constant and the two masses are equal If this had not been true, the two masses would oscillate with different frequencies and the damper would be excited thus taking energy out of the system.

3. Write the equations of motion for the double-pendulum system shown in Fig. 2.43. Assume the displacement angles of the pendulums are small enough to ensure that the spring is always horizontal. The pendulum rods are taken to be massless, of length l, and the springs are attached 3/4 of the way down.

Figure 2.43: Double pendulum



Solution:



Define coordinates

If we write the moment equilibrium about the pivot point of the left pendulem from the free body diagram,

$$M = -mgl\sin\theta_1 - k\frac{3}{4}l\left(\sin\theta_1 - \sin\theta_2\right)\cos\theta_1\frac{3}{4}l = ml^2\ddot{\theta}_1$$

$$ml^2\ddot{\theta}_1 + mgl\sin\theta_1 + \frac{9}{16}kl^2\cos\theta_1\left(\sin\theta_1 - \sin\theta_2\right) = 0$$

Similary we can write the equation of motion for the right pendulem

$$-mgl\sin\theta_2 + k\frac{3}{4}l\left(\sin\theta_1 - \sin\theta_2\right)\cos\theta_2\frac{3}{4}l = ml^2\ddot{\theta}_2$$

As we assumed the angles are small, we can approximate using $\sin\theta_1 \approx \theta_1, \sin\theta_2 \approx \theta_2$, $\cos\theta_1 \approx 1$, and $\cos\theta_2 \approx 1$. Finally the linearized equations of motion becomes,

$$\begin{aligned} ml\ddot{\theta}_1 + mg\theta_1 + \frac{9}{16}kl\left(\theta_1 - \theta_2\right) &= 0 \\ ml\ddot{\theta}_2 + mg\theta_2 + \frac{9}{16}kl\left(\theta_2 - \theta_1\right) &= 0 \end{aligned}$$

Or

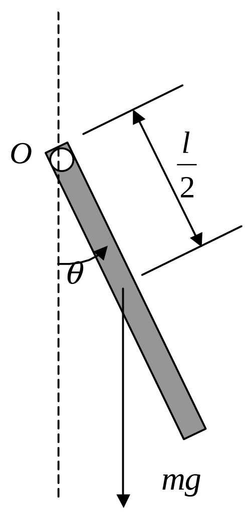
$$\begin{aligned}\ddot{\theta}_1 + \frac{g}{l}\theta_1 + \frac{9}{16}\frac{k}{m}(\theta_1 - \theta_2) &= 0 \\ \ddot{\theta}_2 + \frac{g}{l}\theta_2 + \frac{9}{16}\frac{k}{m}(\theta_2 - \theta_1) &= 0\end{aligned}$$

4. Write the equations of motion of a pendulum consisting of a thin, 2-kg stick of length l suspended from a pivot. How long should the rod be in order for the period to be exactly 1 sec? (The inertia I of a thin stick about an endpoint is $\frac{1}{3}ml^2$. Assume θ is small enough that $\sin\theta \cong \theta$.)

Solution:

Let's use Eq. (2.14)

$$M = I\alpha,$$



Define coordinates and forces

Moment about point O.

$$\begin{aligned}M_O &= -mg \times \frac{l}{2}\sin\theta = I_O\ddot{\theta} \\ &= \frac{1}{3}ml^2\ddot{\theta}\end{aligned}$$

$$\ddot{\theta} + \frac{3g}{2l}\sin\theta = 0$$

As we assumed θ is small,

$$\ddot{\theta} + \frac{3g}{2l}\theta = 0$$

The frequency only depends on the length of the rod

$$\omega^2 = \frac{3g}{2l}$$

$$\begin{aligned} T &= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2l}{3g}} = 2 \\ l &= \frac{3g}{8\pi^2} = 0.3725\,\text{m} \end{aligned}$$

Grandfather clocks have a period of 2 sec, i.e., 1 sec for a swing from one side to the other. This pendulum is shorter because the period is faster. But if the period had been 2 sec, the pendulum length would have been 1.5 meters, and the clock itself would have been about 2 meters to house the pendulum and the clock face.

<Side notes>

(a) Compare the formula for the period, $T = 2\pi\sqrt{\frac{2l}{3g}}$ with the well known formula for the period of a point mass hanging with a string with length l. $T = 2\pi\sqrt{\frac{l}{g}}$.

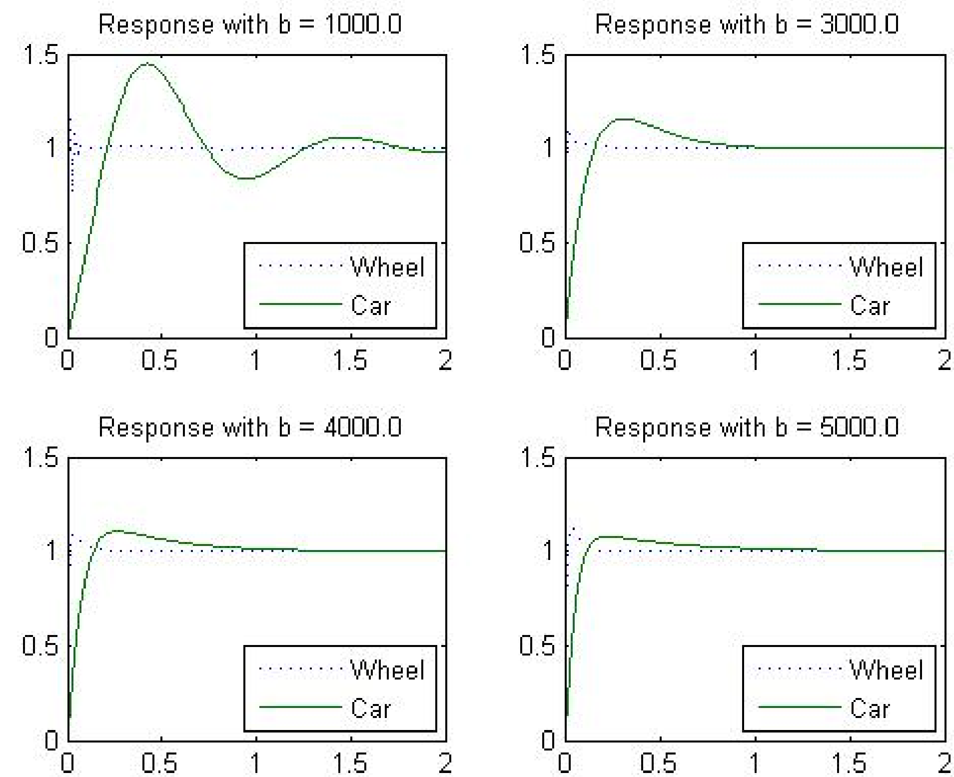
(b) Important!

In general, Eq. (2.14) is valid only when the reference point for the moment and the moment of inertia is the mass center of the body. However, we also can use the formular with a reference point other than mass center when the point of reference is fixed or not accelerating, as was the case here for point O.

5. For the car suspension discussed in Example 2.2, plot the position of the car and the wheel after the car hits a "unit bump" (i.e., r is a unit step) using Matlab. Assume that $m_1 = 10$ kg, $m_2 = 250$ kg, $k_w = 500,000$ N/m, $k_s = 10,000$ N/m. Find the value of b that you would prefer if you were a passenger in the car.

Solution:

The transfer function of the suspension was given in the example in Eq. (2.12) to be:



(a)

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}(s + \frac{k_s}{b})}{s^4 + (\frac{b}{m_1} + \frac{b}{m_2})s^3 + (\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1})s^2 + (\frac{k_w b}{m_1 m_2})s + \frac{k_w k_s}{m_1 m_2}}.$$

This transfer function can be put directly into Matlab along with the numerical values as shown below. Note that b is not the damping ratio, but damping. We need to find the proper order of magnitude for b, which can be done by trial and error. What passengers feel is the position of the car. Some general requirements for the smooth ride will be, slow response with small overshoot and oscillation. While the smallest overshoot is with b=5000, the jump in car position happens the fastest with this damping value.

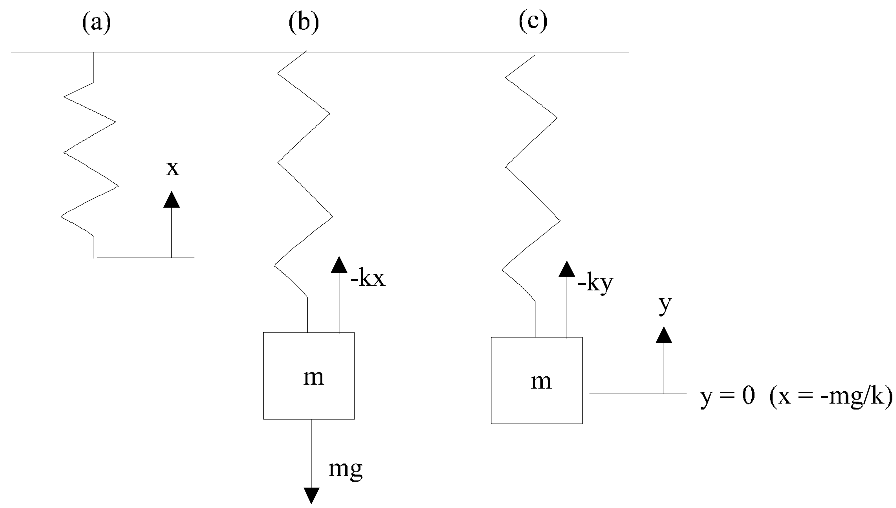
From the figures, $b \approx 3000$ appears to be the best compromise. There is too much overshoot for lower values, and the system gets too fast (and harsh) for larger values.

```
% Problem 2.5
clear all, close all

m1 = 10;
m2 = 250;
kw = 500000;
ks = 10000;
Bd = [ 1000 3000 4000 5000];
t = 0:0.01:2;

for i = 1:4
    b = Bd(i);
    A=[0 1 0 0;-( ks/m1 + kw/m1 ) -b/m1 ks/m1 b/m1;
 0 0 0 1; ks/m2 b/m2 -ks/m2 -b/m2 ];
    B=[0; kw/m1; 0; 0 ];
    C=[ 1 0 0 0; 0 0 1 0 ];
    D=0;
    y=step(A,B,C,D,1,t);
    subplot(2,2,i);
    plot( t, y(:,1), ':', t, y(:,2), '-' );
    legend('Wheel','Car');
    ttl = sprintf('Response with b = %4.1f ',b );
    title(ttl);
end
```

6. Write the equations of motion for a body of mass M suspended from a fixed point by a spring with a constant k. Carefully define where the body's displacement is zero.

Solution:

Some care needs to be taken when the spring is suspended vertically in the presence of the gravity. We define $x = 0$ to be when the spring is unstretched with no mass attached as in (a). The static situation in (b) results from a balance between the gravity force and the spring.



From the free body diagram in (b), the dynamic equation results

$$m\ddot{x} = -kx - mg.$$

We can manipulate the equation

$$m\ddot{x} = -k\left(x + \frac{m}{k}g\right),$$

so if we replace x using $y = x + \frac{m}{k}g$,

$$\begin{aligned} m\ddot{y} &= -ky \\ m\ddot{y} + ky &= 0 \end{aligned}$$

The equilibrium value of x including the effect of gravity is at $x = -\frac{m}{k}g$ and y represents the motion of the mass about that equilibrium point.

An alternate solution method, which is applicable for any problem involving vertical spring motion, is to define the motion to be with respect to the static equilibrium point of the springs including the effect of gravity, and then to proceed as if no gravity was present. In this problem, we would define y to be the motion with respect to the equilibrium point, then the FBD in (c) would result directly in

$$m\ddot{y} = -ky.$$

7. Automobile manufacturers are contemplating building active suspension systems. The simplest change is to make shock absorbers with a changeable damping, $b(u_1)$. It is also possible to make a device to be placed in parallel with the springs that has the ability to supply an equal force, u_2, in opposite directions on the wheel axle and the car body.

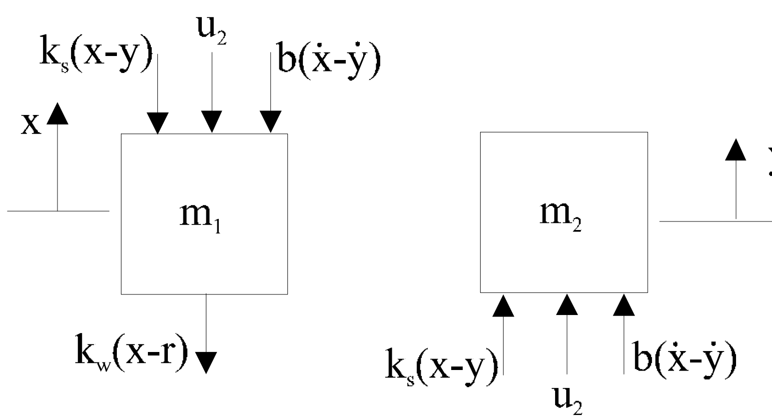
(a) Modify the equations of motion in Example 2.2 to include such control inputs.

(b) Is the resulting system linear?

(c) Is it possible to use the forcer, u_2, to completely replace the springs and shock absorber? Is this a good idea?

Solution:

(a) The FBD shows the addition of the variable force, u_2, and shows b as in the FBD of Fig. 2.5, however, here b is a function of the control variable, u_1. The forces below are drawn in the direction that would result from a positive displacement of x.



Free body diagram

$$
\begin{aligned}
m_1\ddot{x} &= b(u_1)(\dot{y}-\dot{x}) + k_s(y-x) - k_w(x-r) - u_2 \\
m_2\ddot{y} &= -k_s(y-x) - b(u_1)(\dot{y}-\dot{x}) + u_2
\end{aligned}
$$

(b) The system is linear with respect to u_2 because it is additive. But b is not constant so the system is non-linear with respect to u_1 because the control essentially multiplies a state element. So if we add controllable damping, the system becomes non-linear.

(c) It is technically possible. However, it would take very high forces and thus a lot of power and is therefore not done. It is a much better solution to modulate the damping coefficient by changing orifice sizes in the shock absorber and/or by changing the spring forces by increasing or decreasing the pressure in air springs. These features are now available on some cars... where the driver chooses between a soft or stiff ride.

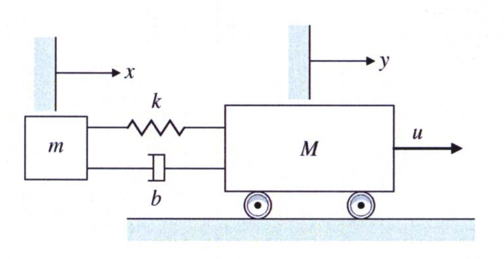
8. In many mechanical positioning systems there is flexibility between one part of the system and another. An example is shown in Figure 2.6

where there is flexibility of the solar panels. Figure 2.44 depicts such a situation, where a force u is applied to the mass M and another mass m is connected to it. The coupling between the objects is often modeled by a spring constant k with a damping coefficient b, although the actual situation is usually much more complicated than this.

(a) Write the equations of motion governing this system.

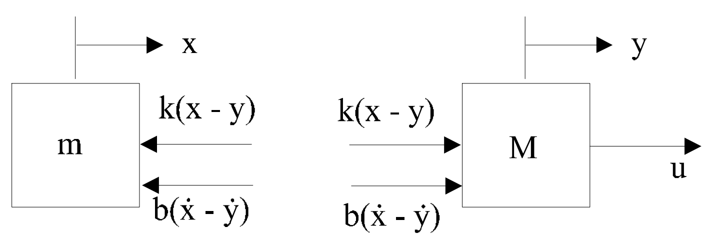
(b) Find the transfer function between the control input, u, and the output, y.

Figure 2.44: Schematic of a system with flexibility



Solution:

(a) The FBD for the system is



Free body diagrams

which results in the equations

$$\begin{aligned} m\ddot{x} &= -k(x-y) - b(\dot{x}-\dot{y}) \\ M\ddot{y} &= u + k(x-y) + b(\dot{x}-\dot{y}) \end{aligned}$$

or

$$\begin{aligned} \ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} - \frac{k}{m}y - \frac{b}{m}\dot{y} &= 0 \\ -\frac{k}{M}x - \frac{b}{M}\dot{x} + \ddot{y} + \frac{k}{M}y + \frac{b}{M}\dot{y} &= \frac{1}{M}u \end{aligned}$$

(b) If we make Laplace Transform of the equations of motion

$$\begin{aligned} s^2X + \frac{k}{m}X + \frac{b}{m}sX - \frac{k}{m}Y - \frac{b}{m}sY &= 0 \\ -\frac{k}{M}X - \frac{b}{M}sX + s^2Y + \frac{k}{M}Y + \frac{b}{M}sY &= \frac{1}{M}U \end{aligned}$$

In matrix form,

$$\begin{bmatrix} ms^2+bs+k & -(bs+k) \\ -(bs+k) & Ms^2+bs+k \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ U \end{bmatrix}$$

From Cramer's Rule,

$$\begin{aligned} Y &= \frac{\det \begin{bmatrix} ms^2+bs+k & 0 \\ -(bs+k) & U \end{bmatrix}}{\det \begin{bmatrix} ms^2+bs+k & -(bs+k) \\ -(bs+k) & Ms^2+bs+k \end{bmatrix}} \\ &= \frac{ms^2+bs+k}{(ms^2+bs+k)(Ms^2+bs+k) - (bs+k)^2}U \end{aligned}$$

Finally,

$$\begin{aligned} \frac{Y}{U} &= \frac{ms^2+bs+k}{(ms^2+bs+k)(Ms^2+bs+k) - (bs+k)^2} \\ &= \frac{ms^2+bs+k}{mMs^4 + (m+M)bs^3 + (M+m)ks^2} \end{aligned}$$

9. Modify the equation of motion for the cruise control in Example 2.1, Eq(2.4), so that it has a control law; that is, let

$$u = K(v_r - v),$$

where

$$\begin{aligned} v_r &= \text{reference speed} \\ K &= \text{constant.} \end{aligned}$$

This is a 'proportional' control law where the difference between v_r and the actual speed is used as a signal to speed the engine up or slow it down. Put the equations in the standard state-variable form with v_r as the input and v as the state. Assume that $m = 1500$ kg and $b = 70$ N · s/ m, and find the response for a unit step in v_r using MATLAB. Using trial and error, find a value of K that you think would result in a control system in which the actual speed converges as quickly as possible to the reference speed with no objectional behavior.

Solution:

$$\dot{v} + \frac{b}{m}v = \frac{1}{m}u$$

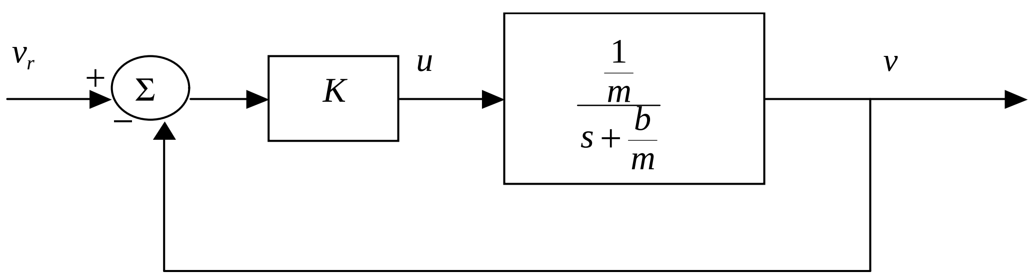
substitute in $u = K(v_r - v)$

$$\dot{v} + \frac{b}{m}v = \frac{1}{m}u = \frac{K}{m}(v_r - v)$$

Rearranging, yields the closed-loop system equations,

$$\dot{v} + \frac{b}{m}v + \frac{K}{m}v = \frac{K}{m}v_r$$

A block diagram of the scheme is shown below where the car dynamics are depicted by its transfer function from Eq. 2.7.



Block diagram

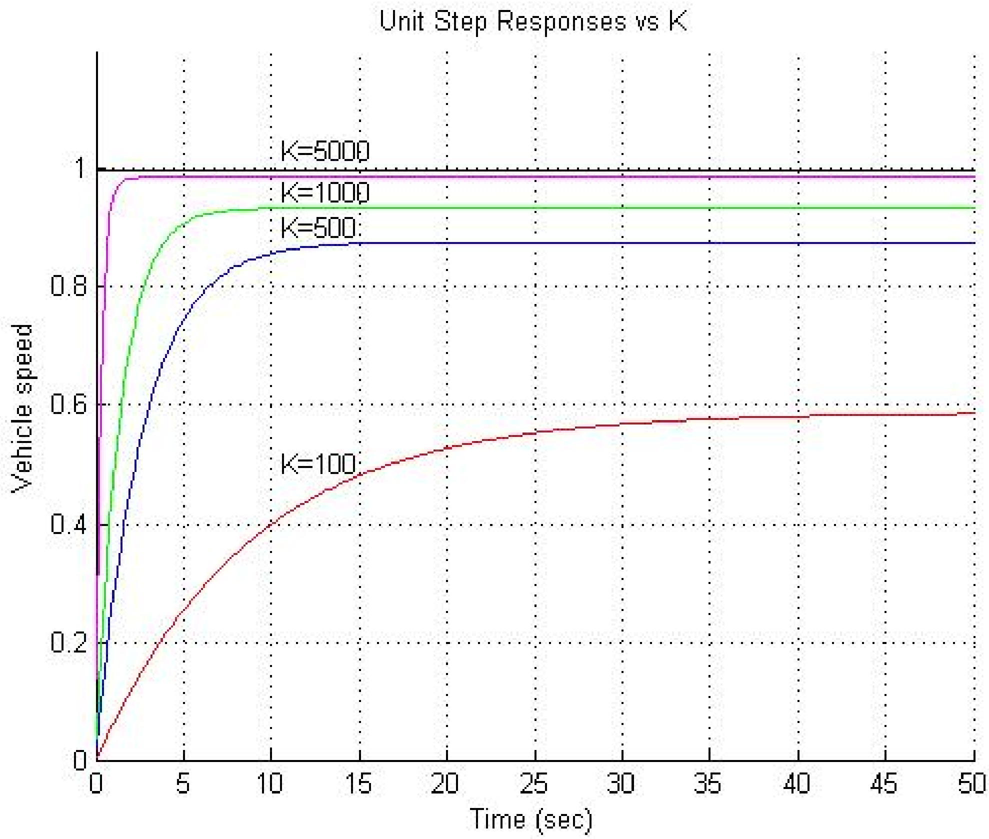
The transfer function of the closed-loop system is,

$$\frac{V(s)}{V_r(s)} = \frac{\frac{K}{m}}{s + \frac{b}{m} + \frac{K}{m}}$$

so that the inputs for Matlab are

$$\begin{aligned} num &= \frac{K}{m} \\ den &= [1 \quad \frac{b}{m} + \frac{K}{m}] \end{aligned}$$

For $K = 100, 500, 1000, 5000$ We have,



We can see that the larger the K is, the better the performance, with no objectionable behaviour for any of the cases. The fact that increasing K also results in the need for higher acceleration is less obvious from the plot but it will limit how fast K can be in the real situation because the engine has only so much poop. Note also that the error with this scheme gets quite large with the lower values of K. You will find out how to eliminate this error in chapter 4 using integral control, which is contained in all cruise control systems in use today. For this problem, a reasonable compromise between speed of response and steady state errors would be $K = 1000$, where it responds in 5 seconds and the steady state error is 5%.

```
% Problem 2.9
clear all, close all

% data
m = 1500;
b = 70;
k = [ 100 500 1000 5000 ];

% Overlay the step response
hold on
t=0:0.2:50;
for i=1:length(k)
    K=k(i);
    num =K/m;
    den = [1  b/m+K/m];
    sys=tf(num,den);
    y = step(sys,t);
    plot(t,y)
end
    hold off
```

Figure 2.45: Robot for delivery of hospital supplies *Source: AP Images*

10. Determine the dynamic equations for lateral motion of the robot in Fig. 2.45. Assume it has 3 wheels with the front a single, steerable wheel where you have direct control of the rate of change of the steering angle, U_{steer}, with geometry as shown in Fig. 2.46. Assume the robot is going in approximately a straight line and its angular deviation from that straight line is very small Also assume that the robot is traveling at a constant speed, V_o. The dynamic equations relating the lateral velocity of the center of the robot as a result of commands in U_{steer} is desired. .

Solution:

This is primarily a problem in kinematics. First, we know that the control input, U_{steer}, is the time rate of change of the steering wheel angle, so

$$\dot{\theta}_s = U_{steer}$$

When θ_s is nonzero, the cart will be turning, so that its orientation wrt the x axis will change at the rate

$$\dot{\Psi} = \frac{V_o \theta_s}{L}.$$

as shown by the diagram below.

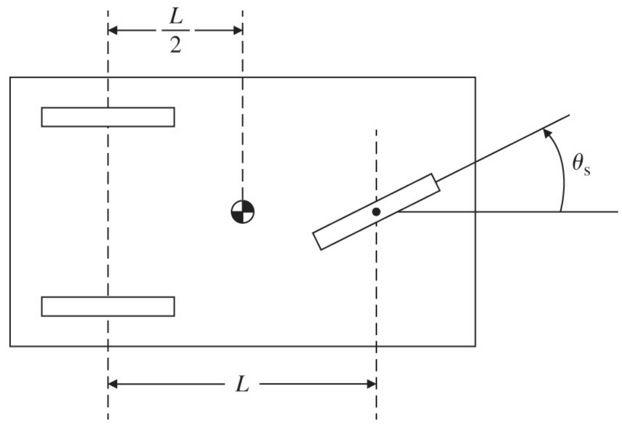


Figure 2.46: Model for robot motion

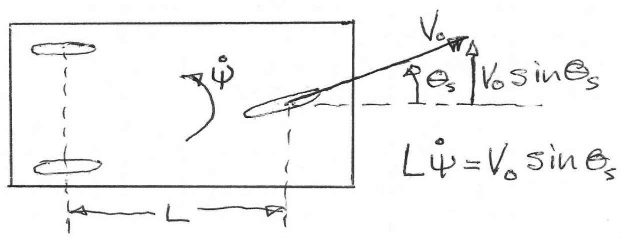
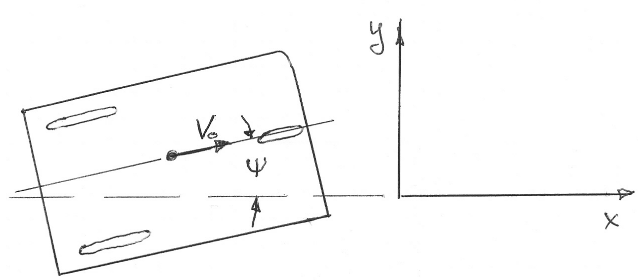


Diagram showing turning rate due to θ_s

The actual change in the carts lateral position will then be proportional to Ψ according to

$$\dot{y} = V_o \Psi$$

as shown below.



Lateral motion as a function of Ψ

These linear equations will hold providing Ψ and θ_s stay small enough that $\sin\Psi \simeq \Psi$, and $\sin\theta_s \simeq \theta_s$.Combining them all, we obtain,

$$\dddot{y} = \frac{V_o^2}{L} U_{steer}$$

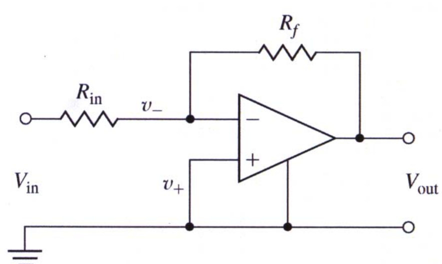
Note that no dynamics come into play here. It was assumed that the velocity is constant and the front wheel angle time rate of change is directly commanded. Therefore, there was no need to invoke Eqs (2.1) or (2.14). As you will see in future chapters, feedback control of such a system with a triple integration is tricky and needs significant damping in the feedback path to achieve stability.

Problems and Solutions for Section 2.2

11. A first step toward a realistic model of an op amp is given by the equations below and shown in Fig. 2.47.

$$\begin{aligned} V_{out} &= \frac{10^7}{s+1}[V_+ - V_-] \\ i_+ &= i_- = 0 \end{aligned}$$

Figure 2.47: Circuit for Problem 11.

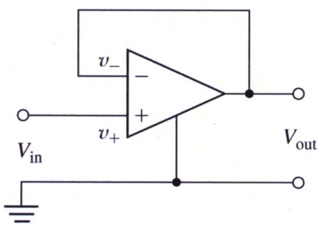


Find the transfer function of the simple amplification circuit shown using this model.

Solution:

As $i_- = 0$,

Figure 2.48: Circuit for Problem 12.



(a)

$$\frac{V_{in} - V_-}{R_{in}} = \frac{V_- - V_{out}}{R_f}$$

$$V_- = \frac{R_f}{R_{in} + R_f} V_{in} + \frac{R_{in}}{R_{in} + R_f} V_{out}$$

$$\begin{aligned} V_{out} &= \frac{10^7}{s+1}[V_+ - V_-] \\ &= \frac{10^7}{s+1}\left(V_+ - \frac{R_f}{R_{in} + R_f} V_{in} - \frac{R_{in}}{R_{in} + R_f} V_{out}\right) \\ &= -\frac{10^7}{s+1}\left(\frac{R_f}{R_{in} + R_f} V_{in} + \frac{R_{in}}{R_{in} + R_f} V_{out}\right) \end{aligned}$$

$$\frac{V_{out}}{V_{in}} = \frac{-10^7 \frac{R_f}{R_{in}+R_f}}{s + 1 + 10^7 \frac{R_{in}}{R_{in}+R_f}}$$

12. Show that the op amp connection shown in Fig. 2.48 results in $V_o = V_{in}$ if the op amp is ideal. Give the transfer function if the op amp has the non-ideal transfer function of Problem 2.11.

Solution:

Ideal case:

$$\begin{aligned} V_{in} &= V_+ \\ V_+ &= V_- \\ V_- &= V_{out} \end{aligned}$$

Non-ideal case:

$$V_{in} = V_+, V_- = V_{out}$$

but,

$$V_+ \neq V_-$$

instead,

$$\begin{aligned} V_{out} &= \frac{10^7}{s+1}[V_+ - V_-] \\ &= \frac{10^7}{s+1}[V_{in} - V_{out}] \end{aligned}$$

so,

$$\frac{V_{out}}{V_{in}} = \frac{\frac{10^7}{s+1}}{1 + \frac{10^7}{s+1}} = \frac{10^7}{s+1+10^7} \cong \frac{10^7}{s+10^7}$$

13. A common connection for a motor power amplifier is shown in Fig. 2.49. The idea is to have the motor current follow the input voltage and the connection is called a current amplifier. Assume that the sense resistor, R_s is very small compared with the feedback resistor, R and find the transfer function from V_{in} to I_a. Also show the transfer function when $R_f = \infty$.

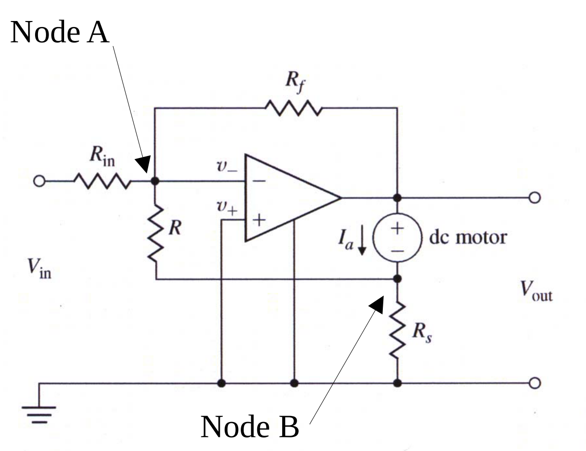


Figure 2.49: Op Amp circuit for Problem 13 with nodes marked.

Solution:

At node A,

$$\frac{V_{in}-0}{R_{in}}+\frac{V_{out}-0}{R_f}+\frac{V_B-0}{R}=0 \tag{93}$$

At node B, with $R_s \ll R$

$$\begin{aligned} I_a+\frac{0-V_B}{R}+\frac{0-V_B}{R_s} &= 0 \\ V_B &= \frac{RR_s}{R+R_s}I_a \\ V_B &\approx R_sI_a \end{aligned} \tag{94}$$

The dynamics of the motor is modeled with negligible inductance as

$$\begin{aligned} J_m\ddot{\theta}_m+b\dot{\theta}_m &= K_tI_a \\ J_ms\Omega+b\Omega &= K_tI_a \end{aligned} \tag{95}$$

At the output, from Eq. 94. Eq. 95 and the motor equation $V_a = I_aR_a+K_es\Omega$

$$\begin{aligned} V_o &= I_aR_s+V_a \\ &= I_aR_s+I_aR_a+K_e\frac{K_tI_a}{J_ms+b} \end{aligned}$$

Substituting this into Eq.93

$$\frac{V_{in}}{R_{in}}+\frac{1}{R_f}\left[I_aR_s+I_aR_a+K_e\frac{K_tI_a}{J_ms+b}\right]+\frac{I_aR_s}{R}=0$$

This expression shows that, in the steady state when $s \to 0$, the current is proportional to the input voltage.

If fact, the current amplifier normally has no feedback from the output voltage, in which case $R_f \to \infty$ and we have simply

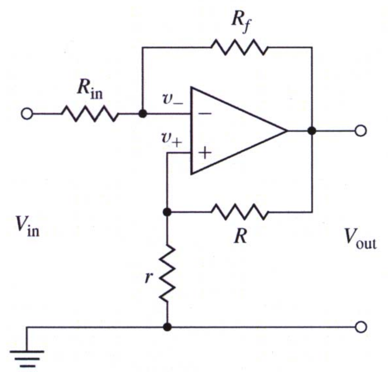
$$\frac{I_a}{V_{in}}=-\frac{R}{R_{in}R_s}$$

14. An op amp connection with feedback to both the negative and the positive terminals is shown in Fig 2.50. If the op amp has the non-ideal transfer function given in Problem 11, give the maximum value possible for the positive feedback ratio, $P=\dfrac{r}{r+R}$ in terms of the negative feedback ratio,$N=\dfrac{R_{in}}{R_{in}+R_f}$ for the circuit to remain stable.

    Solution:

$$\begin{aligned} \frac{V_{in}-V_-}{R_{in}}+\frac{V_{out}-V_-}{R_f} &= 0 \\ \frac{V_{out}-V_+}{R}+\frac{0-V_+}{r} &= 0 \end{aligned}$$

Figure 2.50: Op Amp circuit for Problem 14.



$$
\begin{aligned}
V_- &= \frac{R_f}{R_{in}+R_f}V_{in} + \frac{R_{in}}{R_{in}+R_f}V_{out} \\
&= (1-N)\,V_{in} + NV_{out} \\
V_+ &= \frac{r}{r+R}V_{out} = PV_{out}
\end{aligned}
$$

$$
\begin{aligned}
V_{out} &= \frac{10^7}{s+1}[V_+ - V_-] \\
&= \frac{10^7}{s+1}\left[PV_{out} - (1-N)\,V_{in} - NV_{out}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{V_{out}}{V_{in}} &= \frac{\frac{10^7}{s+1}(1-N)}{\frac{10^7}{s+1}P - \frac{10^7}{s+1}N - 1} \\
&= \frac{10^7\,(1-N)}{10^7P - 10^7N - (s+1)} \\
&= \frac{-10^7\,(1-N)}{s+1-10^7P+10^7N}
\end{aligned}
$$

$$
\begin{aligned}
0 &< 1 - 10^7P + 10^7N \\
P &< N + 10^{-7}
\end{aligned}
$$

15. Write the dynamic equations and find the transfer functions for the circuits shown in Fig. 2.51.

    (a) passive lead circuit
    (b) active lead circuit
    (c) active lag circuit.
    (d) passive notch circuit

Solution:

(a) Passive lead circuit
With the node at y+, summing currents into that node, we get

$$\frac{V_u - V_y}{R_1} + C\frac{d}{dt}(V_u - V_y) - \frac{V_y}{R_2} = 0 \tag{96}$$

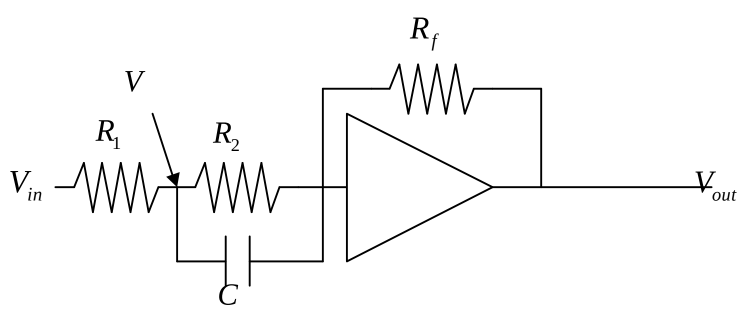
rearranging a bit,

$$C\dot{V}_y + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_y = C\dot{V}_u + \frac{1}{R_1}V_u$$

and, taking the Laplace Transform, we get

$$\frac{V_y(s)}{V_u(s)} = \frac{Cs + \frac{1}{R_1}}{Cs + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

(b) Active lead circuit



Active lead circuit with node marked

$$\frac{V_{in} - V}{R_2} + \frac{0 - V}{R_1} + C\frac{d}{dt}(0 - V) = 0 \tag{97}$$

$$\frac{V_{in} - V}{R_2} = \frac{0 - V_{out}}{R_f} \tag{98}$$

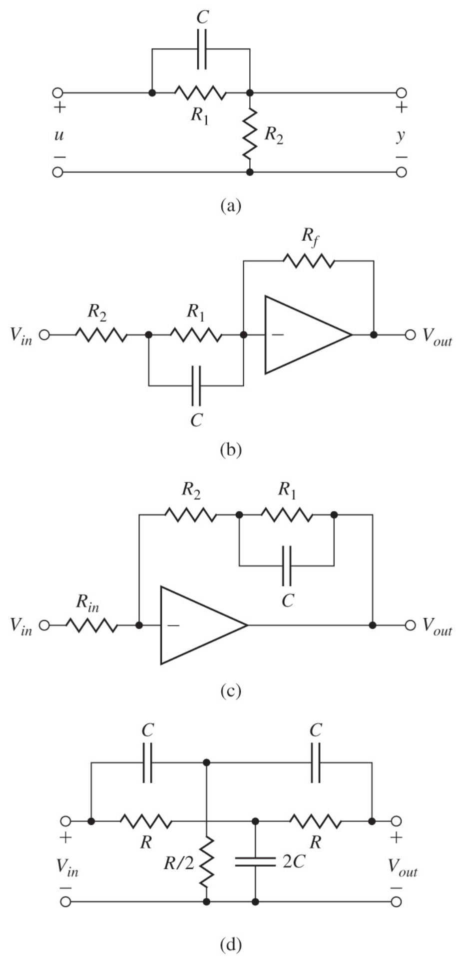


Figure 2.51: (a) Passive lead, (b) active lead, (c) active lag, (d) passive notch circuits

We need to eliminate V. From Eq. 98,

$$V = V_{in} + \frac{R_2}{R_f} V_{out}$$

Substitute V's in Eq. 97.

$$\frac{1}{R_2}\left(V_{in} - V_{in} - \frac{R_2}{R_f}V_{out}\right) - \frac{1}{R_1}\left(V_{in} + \frac{R_2}{R_f}V_{out}\right) - C\left(\dot{V}_{in} + \frac{R_2}{R_f}\dot{V}_{out}\right) = 0$$

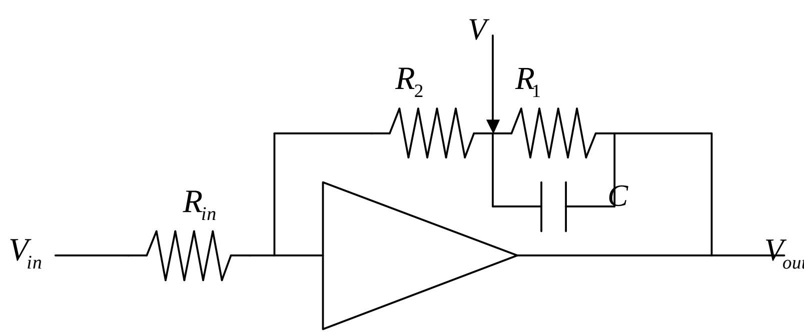
$$\frac{1}{R_1}V_{in} + C\dot{V}_{in} = -\frac{1}{R_f}\left[\left(1 + \frac{R_2}{R_1}\right)V_{out} + R_2 C\dot{V}_{out}\right]$$

Laplace Transform

$$\begin{aligned}\frac{V_{out}}{V_{in}} &= \frac{Cs + \frac{1}{R_1}}{-\frac{1}{R_f}\left(R_2 Cs + 1 + \frac{R_2}{R_1}\right)} \\ &= -\frac{R_f}{R_2}\frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_1 C} + \frac{1}{R_2 C}}\end{aligned}$$

We can see that the pole is at the left side of the zero, which means a lead compensator.

(c) active lag circuit



Active lag circuit with node marked

$$\frac{V_{in} - 0}{R_{in}} = \frac{0 - V}{R_2} = \frac{V - V_{out}}{R_1} + C\frac{d}{dt}(V - V_{out})$$

$$V = -\frac{R_2}{R_{in}}V_{in}$$

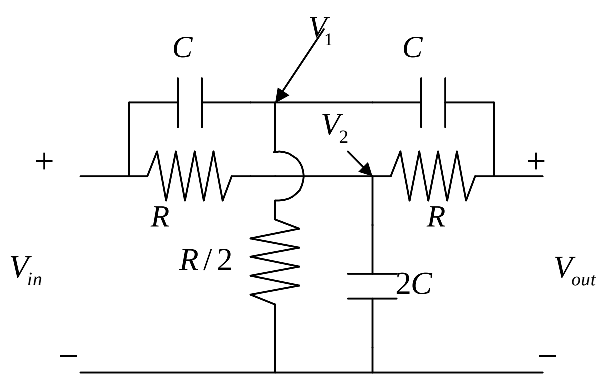
$$\begin{aligned}\frac{V_{in}}{R_{in}} &= \frac{-\frac{R_2}{R_{in}}V_{in} - V_{out}}{R_1} + C\frac{d}{dt}\left(-\frac{R_2}{R_{in}}V_{in} - V_{out}\right) \\ &= \frac{1}{R_1}\left(-\frac{R_2}{R_{in}}V_{in} - V_{out}\right) + C\left(-\frac{R_2}{R_{in}}\dot{V}_{in} - \dot{V}_{out}\right)\end{aligned}$$

$$\frac{1}{R_{in}}\left(1+\frac{R_2}{R_1}\right)V_{in} + \frac{1}{R_{in}}R_2C\dot{V}_{in} = -\frac{1}{R_1}V_{out} - C\dot{V}_{out}$$

$$\begin{aligned}\frac{V_{out}}{V_{in}} &= -\frac{R_1}{R_{in}}\frac{R_2Cs + 1 + \frac{R_2}{R_1}}{R_1Cs+1} \\ &= -\frac{R_2}{R_{in}}\frac{s + \frac{1}{R_2C} + \frac{1}{R_1C}}{s + \frac{1}{R_1C}}\end{aligned}$$

We can see that the pole is at the right side of the zero, which means a lag compensator.

(d) notch circuit



Passive notch filter with nodes marked

$$\begin{aligned}C\frac{d}{dt}(V_{in} - V_1) + \frac{0 - V_1}{R/2} + C\frac{d}{dt}(V_{out} - V_1) &= 0 \\ \frac{V_{in} - V_2}{R} + 2C\frac{d}{dt}(0 - V_2) + \frac{V_{out} - V_2}{R} &= 0 \\ C\frac{d}{dt}(V_1 - V_{out}) + \frac{V_2 - V_{out}}{R} &= 0\end{aligned}$$

We need to eliminat V_1, V_2 from three equations and find the relation between V_{in} and V_{out}

$$\begin{aligned}V_1 &= \frac{Cs}{2\left(Cs + \frac{1}{R}\right)}(V_{in} + V_{out}) \\ V_2 &= \frac{\frac{1}{R}}{2\left(Cs + \frac{1}{R}\right)}(V_{in} + V_{out})\end{aligned}$$

$$\begin{aligned}
& CsV_1 - CsV_{out} + \frac{1}{R}V_2 - \frac{1}{R}V_{out} \\
= \; & Cs\frac{Cs}{2\left(Cs+\frac{1}{R}\right)}\left(V_{in}+V_{out}\right) + \frac{1}{R}\frac{\frac{1}{R}}{2\left(Cs+\frac{1}{R}\right)}\left(V_{in}+V_{out}\right) - \left(Cs+\frac{1}{R}\right)V_{out} \\
= \; & 0
\end{aligned}$$

$$\begin{aligned}
\frac{C^2s^2+\frac{1}{R^2}}{2\left(Cs+\frac{1}{R}\right)}V_{in} &= \left[\left(Cs+\frac{1}{R}\right) - \frac{C^2s^2+\frac{1}{R^2}}{2\left(Cs+\frac{1}{R}\right)}\right]V_{out} \\
\frac{V_{out}}{V_{in}} &= \frac{\frac{C^2s^2+\frac{1}{R^2}}{2\left(Cs+\frac{1}{R}\right)}}{\left(Cs+\frac{1}{R}\right) - \frac{C^2s^2+\frac{1}{R^2}}{2\left(Cs+\frac{1}{R}\right)}} \\
&= \frac{\left(C^2s^2+\frac{1}{R^2}\right)}{2\left(Cs+\frac{1}{R}\right)^2 - \left(C^2s^2+\frac{1}{R^2}\right)} \\
&= \frac{C^2\left(s^2+\frac{1}{R^2C^2}\right)}{C^2s^2+4\frac{Cs}{R}+\frac{1}{R^2}} \\
&= \frac{s^2+\frac{1}{R^2C^2}}{s^2+\frac{4}{RC}s+\frac{1}{R^2C^2}}
\end{aligned}$$

16. The very flexible circuit shown in Fig. 2.52 is called a biquad because its transfer function can be made to be the ratio of two second-order or quadratic polynomials. By selecting different values for R_a, R_b, R_c, and R_d the circuit can realise a low-pass, band-pass, high-pass, or band-reject (notch) filter.

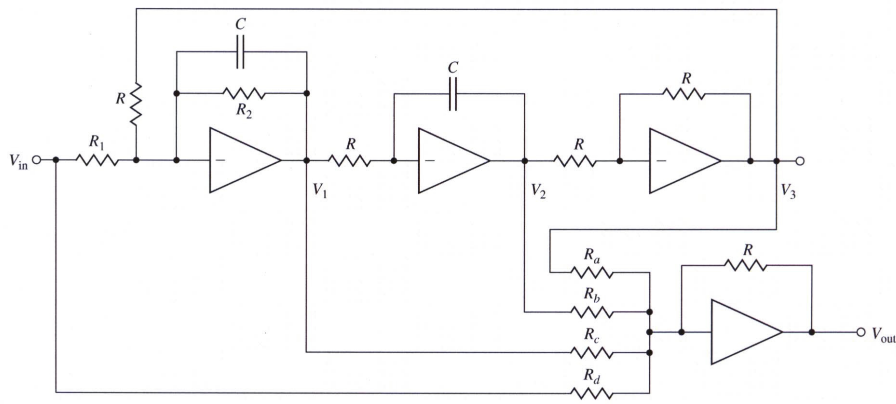
    (a) Show that if $R_a = R$, and $R_b = R_c = R_d = \infty$, the transfer function from V_{in} to V_{out} can be written as the low-pass filter

$$\frac{V_{out}}{V_{in}} = \frac{A}{\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1} \tag{99}$$

    where

$$\begin{aligned}
A &= \frac{R}{R_1} \\
\omega_n &= \frac{1}{RC} \\
\zeta &= \frac{R}{2R_2}
\end{aligned}$$

Figure 2.52: Op-amp biquad



(b) Using the MATLAB comand `step` compute and plot on the same graph the step responses for the biquad of Fig. 2.52 for $A = 2$, $\omega_n = 2$, and $\zeta = 0.1, \quad 0.5$, and 1.0.

Solution:

Before going in to the specific problem, let's find the general form of the transfer function for the circuit.

$$\begin{aligned}
\frac{V_{in}}{R_1} + \frac{V_3}{R} &= -\left(\frac{V_1}{R_2} + C\dot{V}_1\right) \\
\frac{V_1}{R} &= -C\dot{V}_2 \\
V_3 &= -V_2 \\
\frac{V_3}{R_a} + \frac{V_2}{R_b} + \frac{V_1}{R_c} + \frac{V_{in}}{R_d} &= -\frac{V_{out}}{R}
\end{aligned}$$

There are a couple of methods to find the transfer function from V_{in} to V_{out} with set of equations but for this problem, we will directly solve for the values we want along with the Laplace Transform.

From the first three equations, slove for V_1, V_2.

$$\begin{aligned}
\frac{V_{in}}{R_1} + \frac{V_3}{R} &= -\left(\frac{1}{R_2} + Cs\right) V_1 \\
\frac{V_1}{R} &= -CsV_2 \\
V_3 &= -V_2
\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{R_2}+Cs\right)V_1-\frac{1}{R}V_2 &= -\frac{1}{R_1}V_{in}\\ \frac{1}{R}V_1+CsV_2 &= 0\end{aligned}$$

$$\begin{bmatrix}\frac{1}{R_2}+Cs & -\frac{1}{R}\\ \frac{1}{R} & Cs\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix}=\begin{bmatrix}-\frac{1}{R_1}V_{in}\\ 0\end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix}V_1\\ V_2\end{bmatrix} &= \frac{1}{\left(\frac{1}{R_2}+Cs\right)Cs+\frac{1}{R^2}}\begin{bmatrix}Cs & \frac{1}{R}\\ -\frac{1}{R} & \frac{1}{R_2}+Cs\end{bmatrix}\begin{bmatrix}-\frac{1}{R_1}V_{in}\\ 0\end{bmatrix}\\ &= \frac{1}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}\begin{bmatrix}-\frac{C}{R_1}sV_{in}\\ \frac{1}{RR_1}V_{in}\end{bmatrix}\end{aligned}$$

Plug in V_1, V_2 and V_3 to the fourth equation.

$$\begin{aligned}&\frac{V_3}{R_a}+\frac{V_2}{R_b}+\frac{V_1}{R_c}+\frac{V_{in}}{R_d}\\ =\;&\left(-\frac{1}{R_a}+\frac{1}{R_b}\right)V_2+\frac{1}{R_c}V_1+\frac{1}{R_d}V_{in}\\ =\;&\left(-\frac{1}{R_a}+\frac{1}{R_b}\right)\frac{\frac{1}{RR_1}}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}V_{in}+\frac{1}{R_c}\frac{-\frac{C}{R_1}s}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}V_{in}+\frac{1}{R_d}V_{in}\\ =\;&\left[\left(-\frac{1}{R_a}+\frac{1}{R_b}\right)\frac{\frac{1}{RR_1}}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}+\frac{1}{R_c}\frac{-\frac{C}{R_1}s}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}+\frac{1}{R_d}\right]V_{in}\\ =\;&-\frac{V_{out}}{R}\end{aligned}$$

Finally,

$$\begin{aligned}\frac{V_{out}}{V_{in}} &= -R\left[\left(-\frac{1}{R_a}+\frac{1}{R_b}\right)\frac{\frac{1}{RR_1}}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}+\frac{1}{R_c}\frac{-\frac{C}{R_1}s}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}+\frac{1}{R_d}\right]\\ &= -R\frac{\left(-\frac{1}{R_a}+\frac{1}{R_b}\right)\frac{1}{RR_1}-\frac{1}{R_c}\frac{C}{R_1}s+\frac{1}{R_d}\left(C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}\right)}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}\\ &= -\frac{R}{C^2}\frac{\frac{C^2}{R_d}s^2+\left(\frac{1}{R_d}\frac{C}{R_2}-\frac{1}{R_c}\frac{C}{R_1}\right)s+\left(\frac{1}{R_b}-\frac{1}{R_a}\right)\frac{1}{RR_1}+\frac{1}{R_d}\frac{1}{R^2}}{s^2+\frac{1}{R_2C}s+\frac{1}{(RC)^2}}\end{aligned}$$

(a) If $R_a = R,$ and $R_b = R_c = R_d = \infty,$

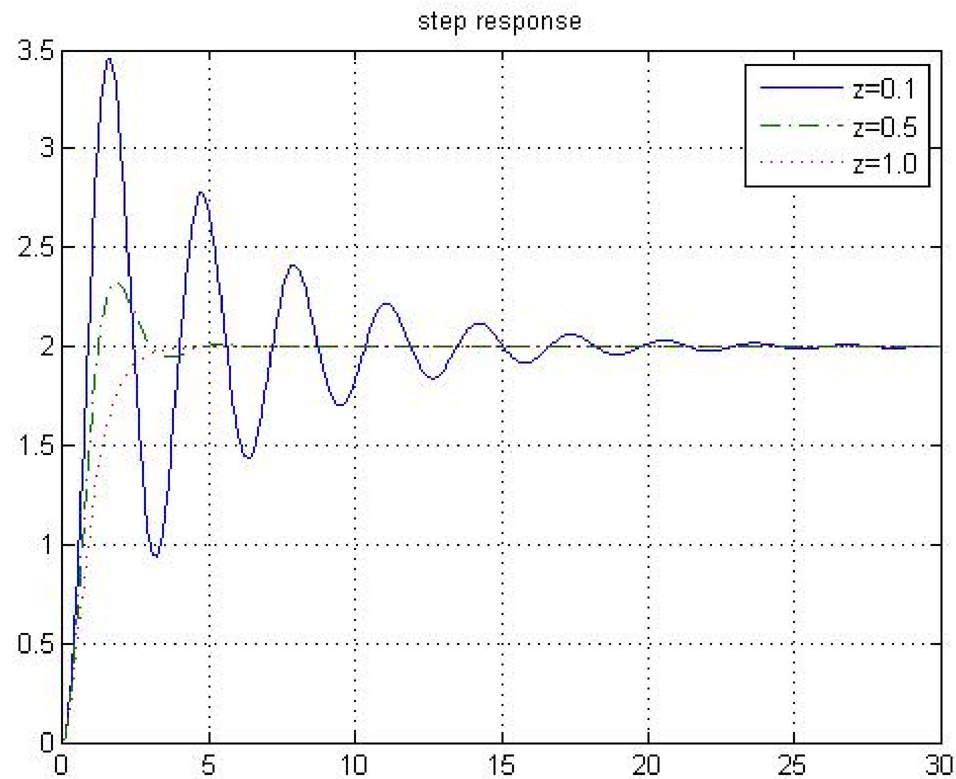
$$
\begin{aligned}
\frac{V_{out}}{V_{in}} &= -\frac{R}{C^2}\frac{\frac{C^2}{R_d}s^2 + \left(\frac{1}{R_d}\frac{C}{R_2} - \frac{1}{R_c}\frac{C}{R_1}\right)s + \left(\frac{1}{R_b} - \frac{1}{R_a}\right)\frac{1}{RR_1} + \frac{1}{R_d}\frac{1}{R^2}}{s^2 + \frac{1}{R_2C}s + \frac{1}{(RC)^2}} \\
&= -\frac{R}{C^2}\frac{-\frac{1}{R}\frac{1}{RR_1}}{s^2 + \frac{1}{R_2C}s + \frac{1}{(RC)^2}} = \frac{\frac{1}{RR_1C^2}}{s^2 + \frac{1}{R_2C}s + \frac{1}{(RC)^2}} \\
&= \frac{\frac{R}{R_1}}{(RC)^2 s^2 + \frac{R^2C}{R_2}s + 1}
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{R}{R_1} &= A \\
(RC)^2 &= \frac{1}{\omega_n^2} \\
2\frac{\zeta}{\omega_n} &= \frac{R^2C}{R_2}
\end{aligned}
$$

$$
\begin{aligned}
\omega_n &= \frac{1}{RC} \\
\zeta &= \frac{\omega_n}{2}\frac{R^2C}{R_2} = \frac{1}{2RC}\frac{R^2C}{R_2} = \frac{R}{2R_2}
\end{aligned}
$$

(b) Step response using MatLab



Step responses

```
% Problem 2.16
A = 2;
wn = 2;
z = [ 0.1 0.5 1.0 ];

hold on
for i = 1:3
    num = [ A ];
    den = [ 1/wn^2 2*z(i)/wn 1 ]
    step( num, den )
end
hold off
```

17. Find the equations and transfer function for the biquad circuit of Fig. 2.52 if $R_a = R$, $R_d = R_1$ and $R_b = R_c = \infty$.

Solution:

$$
\begin{aligned}
\frac{V_{out}}{V_{in}} &= -\frac{R}{C^2}\frac{\frac{C^2}{R_d}s^2+\left(\frac{1}{R_d}\frac{C}{R_2}-\frac{1}{R_c}\frac{C}{R_1}\right)s+\left(\frac{1}{R_b}-\frac{1}{R_a}\right)\frac{1}{RR_1}+\frac{1}{R_d}\frac{1}{R^2}}{s^2+\frac{1}{R_2C}s+\frac{1}{(RC)^2}} \\
&= -\frac{R}{C^2}\frac{\frac{C^2}{R_1}s^2+\left(\frac{1}{R_1}\frac{C}{R_2}\right)s+\left(-\frac{1}{R}\right)\frac{1}{RR_1}+\frac{1}{R_1}\frac{1}{R^2}}{s^2+\frac{1}{R_2C}s+\frac{1}{(RC)^2}} \\
&= -\frac{R}{R_1}\frac{s^2+\frac{1}{R_2C}s}{s^2+\frac{1}{R_2C}s+\frac{1}{(RC)^2}}
\end{aligned}
$$

Problems and Solutions for Section 2.3

18. The torque constant of a motor is the ratio of torque to current and is often given in ounce-inches per ampere. (ounce-inches have dimension force-distance where an ounce is 1/16 of a pound.) The electric constant of a motor is the ratio of back emf to speed and is often given in volts per 1000 rpm. In consistent units the two constants are the same for a given motor.

    (a) Show that the units ounce-inches per ampere are proportional to volts per 1000 rpm by reducing both to MKS (SI) units.

    (b) A certain motor has a back emf of 25 V at 1000 rpm. What is its torque constant in ounce-inches per ampere?

    (c) What is the torque constant of the motor of part (b) in newton-meters per ampere?

Solution:

Before going into the problem, let's review the units.

- Some remarks on non SI units.
  - Ounce
    $$1\text{oz} = 2.835 \times 10^{-2}\,\text{kg}$$
    Actuall, the ounce is a unit of mass, but like pounds, it is commonly used as a unit of force. If we translate it as force,
    $$1\text{oz(f)} = 2.835 \times 10^{-2}\ \text{kgf} = 2.835 \times 10^{-2} \times 9.81\,\text{N} = 0.2778\,\text{N}$$
  - Inch
    $$1\,\text{in} = 2.540 \times 10^{-2}\,\text{m}$$
  - RPM (Revolution per Minute)
    $$1\ \text{RPM} = \frac{2\pi\ \text{rad}}{60\ \text{s}} = \frac{\pi}{30}\ \text{rad/s}$$
- Relation between SI units
  - Voltage and Current
    $$\begin{aligned} Volts \cdot Current(amps) &= Power = Energy(joules)/\sec \\ Volts &= \frac{Joules/\sec}{amps} = \frac{Newton - meters/\sec}{amps} \end{aligned}$$

(a) Relation between torque constant and electric constant.
Torque constant:

$$\frac{1 \text{ ounce} \times 1 \text{ inch}}{1 \text{ Ampere}} = \frac{0.2778\,\text{N} \times 2.540 \times 10^{-2}\,\text{m}}{1\,\text{A}} = 7.056{\times}10^{-3}\,\text{N m/ A}$$

Electric constant:

$$\frac{1\,\text{V}}{1000 \text{ RPM}} = \frac{1\,\text{J/(A sec)}}{1000 \times \frac{\pi}{30} \text{ rad/ s}} = 9.549 \times 10^{-3}\,\text{N m/ A}$$

So,

$$\begin{aligned} 1 \text{ oz in/ A} &= \frac{7.056 \times 10^{-3}}{9.549 \times 10^{-3}} \text{ V/1000 RPM} \\ &= (0.739) \text{ V/1000 RPM} \end{aligned}$$

and the constant of proportionality $= (0.739)$.

(b)

$$25\,\text{V/1000 RPM} = 25 \times \frac{1}{0.739} \text{ oz in/ A} = 33.8 \text{ oz in/ A}$$

(c)

$$25\,\text{V/1000 RPM} = 25 \times 9.549 \times 10^{-3}\,\text{N m/ A} = 0.239\,\text{N m/ A}$$

19. The electromechanical system shown in Fig. 2.53 represents a simplified model of a capacitor microphone. The system consists in part of a parallel plate capacitor connected into an electric circuit. Capacitor plate a is rigidly fastened to the microphone frame. Sound waves pass through the mouthpiece and exert a force $f_s(t)$ on plate b, which has mass M and is connected to the frame by a set of springs and dampers. The capacitance C is a function of the distance x between the plates, as follows:

$$C(x) = \frac{\varepsilon A}{x},$$

where

$$\begin{aligned} \varepsilon &= \text{dielectric constant of the material between the plates,} \\ A &= \text{surface area of the plates.} \end{aligned}$$

The charge q and the voltage e across the plates are related by

$$q = C(x)e.$$

The electric field in turn produces the following force f_e on the movable plate that opposes its motion:

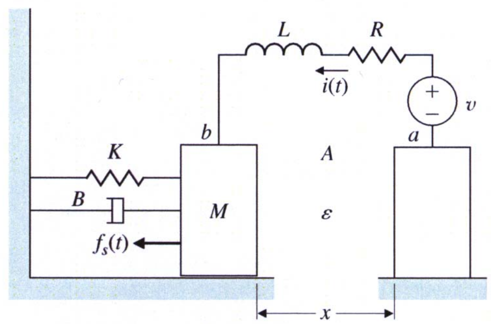
$$f_e = \frac{q^2}{2\varepsilon A}$$

(a) Write differential equations that describe the operation of this system. (It is acceptable to leave in nonlinear form.)

(b) Can one get a linear model?

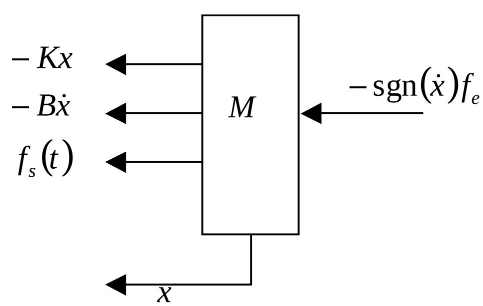
(c) What is the output of the system?

Figure 2.53: Simplified model for capacitor microphone



Solution:

(a) The free body diagram of the capacitor plate b



Free body diagram

So the equation of motion for the plate is

$$M\ddot{x} + B\dot{x} + Kx + f_e sgn(\dot{x}) = f_s(t).$$

The equation of motion for the circuit is

$$v = iR + L\frac{d}{dt}i + e$$

where e is the voltage across the capacitor,

$$e = \frac{1}{C}\int i(t)dt$$

and where $C = \varepsilon A/x$, a variable. Because $i = \frac{d}{dt}q$ and $e = q/C$, we can rewrite the circuit equation as

$$v = R\dot{q} + L\ddot{q} + \frac{qx}{\varepsilon A}$$

In summary, we have these two, couptled, non-linear differential equation.

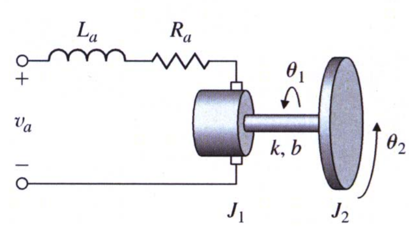
$$\begin{aligned} M\ddot{x} + b\dot{x} + kx + \text{sgn}(\dot{x})\frac{q^2}{2\varepsilon A} &= f_s(t) \\ R\dot{q} + L\ddot{q} + \frac{qx}{\varepsilon A} &= v \end{aligned}$$

(b) The sgn function, q^2, and qx, terms make it impossible to determine a useful linearized version.

(c) The signal representing the voice input is the current, i, or $\dot{q}$.

20. A very typical problem of electromechanical position control is an electric motor driving a load that has one dominant vibration mode. The problem arises in computer-disk-head control, reel-to-reel tape drives, and many other applications. A schematic diagram is sketched in Fig. 2.54. The motor has an electrical constant K_e, a torque constant K_t, an armature inductance L_a, and a resistance R_a. The rotor has an inertia J_1 and a viscous friction B. The load has an inertia J_2. The two inertias are connected by a shaft with a spring constant k and an equivalent viscous damping b. Write the equations of motion.

Figure 2.54: Motor with a flexible load



(a)

Solution:

(a) Rotor:

$$J_1\ddot{\theta}_1 = -B\dot{\theta}_1 - b\left(\dot{\theta}_1 - \dot{\theta}_2\right) - k(\theta_1 - \theta_2) + T_m$$

Load:

$$J_2\ddot{\theta}_2 = -b\left(\dot{\theta}_2 - \dot{\theta}_1\right) - k\left(\theta_2 - \theta_1\right)$$

Circuit:

$$v_a - K_e\dot{\theta}_1 = L_a\frac{d}{dt}i_a + R_a i_a$$

Relation between the output torque and the armature current:

$$T_m = K_t i_a$$

21. For the robot in Fig. 2.45, assume you have command of the torque on a servo motor that is connected to the drive wheels with gears that have a 2:1 ratio so that the torque on the wheels is increased by a factor of 2 over that delivered by the servo. Determine the dynamic equations relating the speed of the robot with respect to the torque command of the servo. Your equations will require certain quantities, e.g., mass of vehicle, inertia and radisus of the wheels, etc. Assume you have access to whatever you

Fig. 2.45 Hospital robot

need. .

(a) **Solution:** First, let's consider the problem for the case along the lines of the development in Section 2.3.3. That is, a system where the torque is applied by a motor on a gear that is simply accelerating an attached gear, like the picture in Fig. 2.35(b). This basically is assuming that the robot has no mass; but we'll come back to that. In order to multiply the torque by a factor of 2, the motor must have a gear that is half the size of the gear attached to the wheel, i.e., $n = 2$ in Eq. (2.78). For simplicity, let's also assume there is no damping on the motor shaft or the wheel shaft, so b_1 and b_2 are both $= 0$. If the wheel was not attached to the robot, Eq. (2.78) yields

$$(J_w + J_m n^2)\ddot{\theta}_w = nT_m$$

where $J_w =$ the inertia of the drive wheel, $J_m =$ motor inertia, $\ddot{\theta}_w =$ wheel angular acceleration, $n = 2$, and $T_m =$ commanded torque from the motor. However, the mass of the robot plus all it's wheels need to be taken into account, since the acceleration of the drive wheel is directly related to the acceleration of the robot and its other wheels provided there is no slippage. (And, hospital robots probably won't be burning rubber). So that means we need to add the rotational inertia of the two other wheels and the inertia due to the translation of the cart plus the center of mass of the 3 wheels. The acceleration of all these quantities are directly related through kinematics because of the nonslip assumption. Let's assume the other two wheels have the same radius as the drive wheel; therefore, their angular acceleration is also $\ddot{\theta}_w$ and we'll also assume they have the same inertia as the drive wheel. That means, if we neglect the translation inertia of the system, the equation becomes

$$(3J_w + J_m n^2)\ddot{\theta}_w = nT_m$$

When you apply a torque to a drive wheel, that torque partly provides an angular accelation of the wheel and the remainder is tranferred to the contact point as a friction force that accelerates the mass of the vehicle. That friction force is

$$f = m_{tot}a = m_{tot}r_w\ddot{\theta}_w$$

where m_{tot} = the mass of the cart plus all three wheels. By looking at a FBD of the wheel, we see that the friction force acts as a torque ($= r_w f$) applied to the wheel; and, therefore, it is essentially another angular inertia term in the equation above. So the end result is:

$$\begin{aligned}(m_{tot}r_w^2 + 3J_w + J_m n^2)\ddot{\theta}_w &= nT_m \\ (m_{tot}r_w^2 + 3J_w + 4J_m)\ddot{\theta}_w &= 2T_m\end{aligned}$$

22. Using Fig. 2.35, derive the transfer function between the applied torque, T_m, and the output, θ_2, for the case when there is a spring attached to the output load. That is, there is a torque applied to the output load, T_s, where $T_s = -K_s\theta_2$

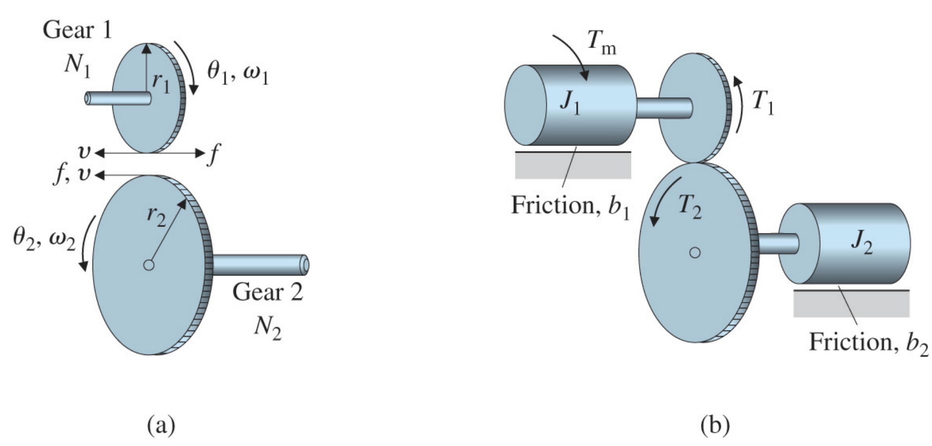


Fig. 2.35 (a) geometry definitions and forces on teeth (b) definitions for the dynamic analysis.

Solution: Equation (2.78), repeated, is

$$(J_2 + J_1 n^2)\ddot{\theta}_2 + (b_2 + b_1 n^2)\dot{\theta}_2 = nT_m$$

Since the spring is only applied to the second rotational mass, its torque only effects Eq. (2.77). Adding the spring torque to Eq. 2.77 yields

$$J_2\ddot{\theta}_2 + b_2\dot{\theta}_2 + K_s\theta_2 = T_2$$

and following the devleopment in the text on page 57, we see that the result is a revised version of Eq. (2.78), that is

$$(J_2 + J_1 n^2)\ddot{\theta}_2 + (b_2 + b_1 n^2)\dot{\theta}_2 + K_s\theta_2 = nT_m$$

Problems and Solutions for Section 2.4

23. A precision-table leveling scheme shown in Fig. 2.57 relies on thermal expansion of actuators under two corners to level the table by raising or lowering their respective corners. The parameters are:

$$\begin{aligned} T_{\text{act}} &= \text{actuator temperature}, \\ T_{\text{amb}} &= \text{ambient air temperature}, \\ R_f &= \text{heat} - \text{flow coefficient between the actuator and the air}, \\ C &= \text{thermal capacity of the actuator}, \\ R &= \text{resistance of the heater}. \end{aligned}$$

Assume that (1) the actuator acts as a pure electric resistance, (2) the heat flow into the actuator is proportional to the electric power input, and (3) the motion d is proportional to the difference between T_{act} and T_{amb} due to thermal expansion. Find the differential equations relating the height of the actuator d versus the applied voltage v_i.

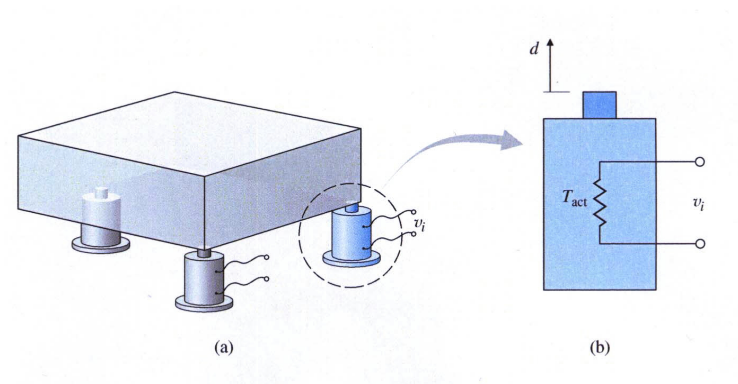


Fig. 2.57 (a) Precision table kept level by actuators; (b) side view of one actuator

Solution:

Electric power in is proportional to the heat flow in

$$\dot{Q}_{in} = K_q \frac{v_i^2}{R}$$

and the heat flow out is from heat transfer to the ambient air

$$\dot{Q}_{out} = \frac{1}{R_f}\left(T_{act} - T_{amb}\right).$$

The temperature is governed by the difference in heat flows

$$\begin{aligned} \dot{T}_{act} &= \frac{1}{C}\left(\dot{Q}_{in} - \dot{Q}_{out}\right) \\ &= \frac{1}{C}\left(K_q \frac{v_i^2}{R} - \frac{1}{R_f}\left(T_{act} - T_{amb}\right)\right) \end{aligned}$$

and the actuator displacement is

$$d = K\left(T_{act} - T_{amb}\right).$$

where T_{amb} is a given function of time, most likely a constant for a table inside a room. The system input is v_i and the system output is d.

24. An air conditioner supplies cold air at the same temperature to each room on the fourth floor of the high-rise building shown in Fig. 2.58(a). The floor plan is shown in Fig. 2.58(b). The cold air flow produces an equal amount of heat flow q out of each room. Write a set of differential equations governing the temperature in each room, where

$$\begin{aligned} T_o &= \text{temperature outside the building}, \\ R_o &= \text{resistance to heat flow through the outer walls}, \\ R_i &= \text{resistance to heat flow through the inner walls}. \end{aligned}$$

Assume that (1) all rooms are perfect squares, (2) there is no heat flow through the floors or ceilings, and (3) the temperature in each room is uniform throughout the room. Take advantage of symmetry to reduce the number of differential equations to three.

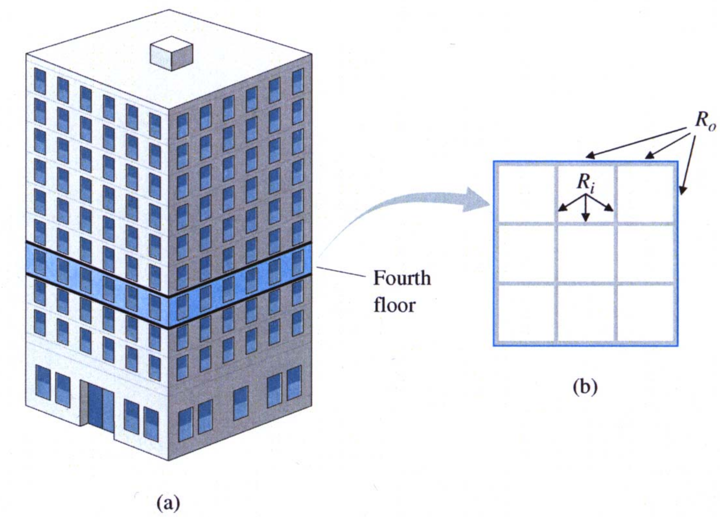


Fig. 2.58 Building air conditioning: (a) high-rise building, (b) floor plan of the fourth floor

Solution:

We can classify 9 rooms to 3 types by the number of outer walls they have.

Type 1	Type 2	Type 1
Type 2	Type 3	Type 2
Type 1	Type 2	Type 1

We can expect the hotest rooms on the outside and the corners hotest of all, but solving the equations would confirm this intuitive result. That is,

$$T_o > T_1 > T_2 > T_3$$

and, with a same cold air flow into every room, the ones with some sun load will be hotest.

Let's redefince the resistances

$$R_o = \text{resistance to heat flow through one unit of outer wall}$$
$$R_i = \text{resistance to heat flow through one unit of inner wall}$$

Room type 1:

$$q_{out} = \frac{2}{R_i}(T_1 - T_2) + q$$
$$q_{in} = \frac{2}{R_o}(T_o - T_1)$$

$$\begin{aligned}\dot{T}_1 &= \frac{1}{C}(q_{in} - q_{out}) \\ &= \frac{1}{C}\left[\frac{2}{R_o}(T_o - T_1) - \frac{2}{R_i}(T_1 - T_2) - q\right]\end{aligned}$$

Room type 2:

$$\begin{aligned}q_{in} &= \frac{1}{R_o}(T_o - T_2) + \frac{2}{R_i}(T_1 - T_2) \\ q_{out} &= \frac{1}{R_i}(T_2 - T_3) + q\end{aligned}$$

$$\dot{T}_2 = \frac{1}{C}\left[\frac{1}{R_o}(T_o - T_2) + \frac{2}{R_i}(T_1 - T_2) - \frac{1}{R_i}(T_2 - T_3) - q\right]$$

Room type 3:

$$\begin{aligned}q_{in} &= \frac{4}{R_i}(T_2 - T_3) \\ q_{out} &= q\end{aligned}$$

$$\dot{T}_3 = \frac{1}{C}\left[\frac{4}{R_i}(T_2 - T_3) - q\right]$$

25. For the two-tank fluid-flow system shown in Fig. 2.59, find the differential equations relating the flow into the first tank to the flow out of the second

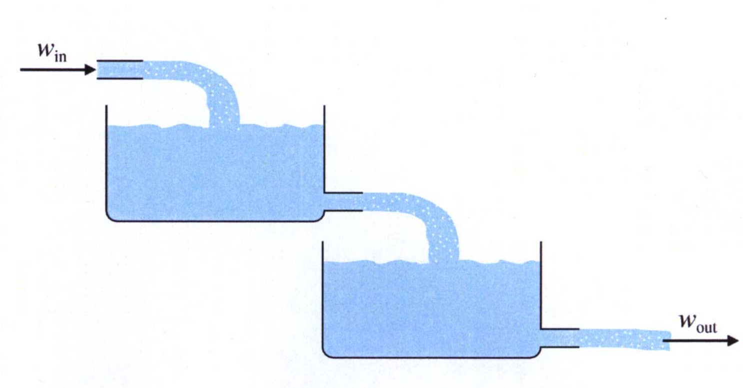


Fig. 2.59 Two-tank fluid-flow system for Problem 25

tank.

Solution:

This is a variation on the problem solved in Example 2.19 and the definitions of terms is taken from that. From the relation between the height of the water and mass flow rate, the continuity equations are

$$\begin{aligned}\dot{m}_1 &= \rho A_1 \dot{h}_1 = w_{in} - w \\ \dot{m}_2 &= \rho A_2 \dot{h}_2 = w - w_{out}\end{aligned}$$

Also from the relation between the pressure and outgoing mass flow rate,

$$\begin{aligned}w &= \frac{1}{R_1}(\rho g h_1)^{\frac{1}{2}} \\ w_{out} &= \frac{1}{R_2}(\rho g h_2)^{\frac{1}{2}}\end{aligned}$$

Finally,

$$\begin{aligned}\dot{h}_1 &= -\frac{1}{\rho A_1 R_1}(\rho g h_1)^{\frac{1}{2}} + \frac{1}{\rho A_1} w_{in} \\ \dot{h}_2 &= \frac{1}{\rho A_2 R_1}(\rho g h_1)^{\frac{1}{2}} - \frac{1}{\rho A_2 R_2}(\rho g h_2)^{\frac{1}{2}}.\end{aligned}$$

26. A laboratory experiment in the flow of water through two tanks is sketched in Fig. 2.60. Assume that Eq. (2.93) describes flow through the equal-sized holes at points A, B, or C.

    (a) With holes at B and C but none at A, write the equations of motion for this system in terms of h_1 and h_2. Assume that when $h_2 = 10$ cm, the outflow is 200 g/min.

    (b) At $h_1 = 30$ cm and $h_2 = 10$ cm, compute a linearized model and the transfer function from pump flow (in cubic centimeters per minute) to h_2.

    (c) Repeat parts (a) and (b) assuming hole B is closed and hole A is open. Assume that $h_3 = 20$ cm, $h_1 > 20$ cm, and $h_2 < 20$.cm.

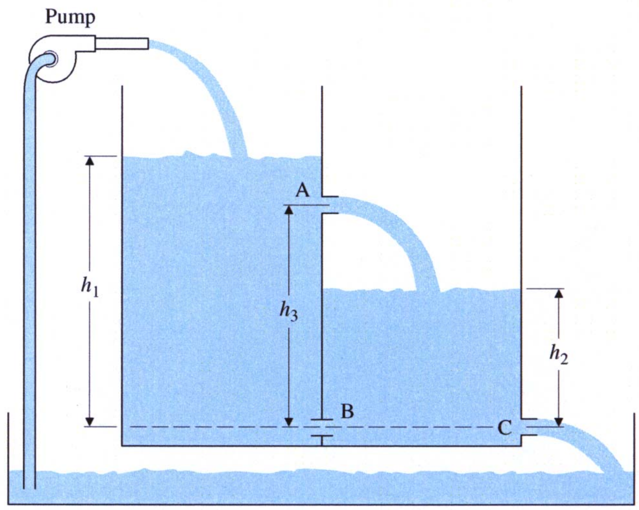


Fig. 2.60 Two-tank fluid-flow system for Problem 26

Solution:

(a) Following the solution of Example 2.19, and assuming the area of both tanks is A, the values given for the heights ensure that the water will flow according to

$$\begin{aligned} W_B &= \frac{1}{R}\left[\rho g\left(h_1 - h_2\right)\right]^{\frac{1}{2}} \\ W_C &= \frac{1}{R}\left[\rho g h_2\right]^{\frac{1}{2}} \\ W_B - W_C &= \rho A\dot{h}_2 \\ W_{in} - W_B &= \rho A\dot{h}_1 \end{aligned}$$

From the outflow information given, we can compute the orifice resistance, R, noting that for water, $\rho = 1$ gram/cc and $g = 981$ cm/sec$^2 \simeq 1000$ cm/sec^2.

$$\begin{aligned} W_C &= 200\,\mathrm{g/\,mn} = \frac{1}{R}\sqrt{\rho g h_2} = \frac{1}{R}\sqrt{\rho g \times 10\,\mathrm{cm}} \\ R &= \frac{\sqrt{\rho g \times 10\,\mathrm{cm}}}{200\,\mathrm{g/\,mn}} = \frac{\sqrt{1\,\mathrm{g/\,cm^3} \times 1000\,\mathrm{cm/\,s^2} \times 10\,\mathrm{cm}}}{200\,\mathrm{g}/60\,\mathrm{s}} \\ &= \frac{100}{200}60\sqrt{\frac{\mathrm{g\,cm^2\,s^2}}{\mathrm{cm^3\,s^2\,g^2}}} = 30\,\mathrm{g}^{-\frac{1}{2}}\,\mathrm{cm}^{-\frac{1}{2}} \end{aligned}$$

(b) The nonlinear equations from above are

$$\begin{aligned}
\dot{h}_1 &= -\frac{1}{\rho AR}\sqrt{\rho g\left(h_1 - h_2\right)} + \frac{1}{\rho A}W_{in} \\
\dot{h}_2 &= \frac{1}{\rho AR}\sqrt{\rho g\left(h_1 - h_2\right)} - \frac{1}{\rho AR}\sqrt{\rho g h_2}
\end{aligned}$$

The square root functions need to be linearized about the nominal heights. In general the square root function can be linearized as below

$$\begin{aligned}
\sqrt{x_0 + \delta x} &= \sqrt{x_0\left(1 + \frac{\delta x}{x_0}\right)} \\
&\cong \sqrt{x_0}\left(1 + \frac{1}{2}\frac{\delta x}{x_0}\right)
\end{aligned}$$

So let's assume that $h_1 = h_{10} + \delta h_1$ and $h_2 = h_{20} + \delta h_2$ where $h_{10} = 30$ cm and $h_{20} = 10$ cm. And for round numbers, let's assume the area of each tank $A = 100$ cm^2. The equations above then reduce to

$$\begin{aligned}
\delta\dot{h}_1 &= -\frac{1}{(1)(100)(30)}\sqrt{(1)(1000)\left(30 + \delta h_1 - 10 - \delta h_2\right)} + \frac{1}{(1)(100)}W_{in} \\
\delta\dot{h}_2 &= \frac{1}{(1)(100)(30)}\sqrt{(1)(1000)\left(30 + \delta h_1 - 10 - \delta h_2\right)} - \frac{1}{(1)(100)(30)}\sqrt{(1)(1000)(10 + \delta h_2)}
\end{aligned}$$

which, with the square root approximations, is equivalent to,

$$\begin{aligned}
\delta\dot{h}_1 &= -\frac{\sqrt{2}}{30}(1 + \frac{1}{40}\delta h_1 - \frac{1}{40}\delta h_2) + \frac{1}{100}W_{in} \\
\delta\dot{h}_2 &= \frac{\sqrt{2}}{30}(1 + \frac{1}{40}\delta h_1 - \frac{1}{40}\delta h_2) - \frac{1}{30}(1 + \frac{1}{20}\delta h_2)
\end{aligned}$$

The nominal inflow $W_{nom} = \frac{10}{3}\sqrt{2}$cc/sec is required in order for the system to be in equilibrium, as can be seen from the first equation. So we will define the total inflow to be $W_{in} = W_{nom} + \delta W$. Including the nominal inflow, the equations become

$$\begin{aligned}
\delta\dot{h}_1 &= -\frac{\sqrt{2}}{1200}(\delta h_1 - \delta h_2) + \frac{1}{100}\delta W \\
\delta\dot{h}_2 &= \frac{\sqrt{2}}{1200}\delta h_1 + (\frac{\sqrt{2}}{1200} - \frac{1}{600})\delta h_2 + \frac{\sqrt{2} - 1}{30}
\end{aligned}$$

However, holding the nominal flow rate maintains h_1 at equilibrium, but h_2 will not stay at equilibrium. Instead, there will be a constant term increasing h_2. Thus the standard transfer function will not result.

(c) With hole B closed and hole A open, the relevant relations are

$$\begin{aligned} W_{in} - W_A &= \rho A \dot{h}_1 \\ W_A &= \frac{1}{R}\sqrt{\rho g(h_1 - h_3)} \\ W_A - W_C &= \rho A \dot{h}_2 \\ W_C &= \frac{1}{R}\sqrt{\rho g h_2} \end{aligned}$$

$$\begin{aligned} \dot{h}_1 &= -\frac{1}{\rho A R}\sqrt{\rho g(h_1 - h_3)} + \frac{1}{\rho A} W_{in} \\ \dot{h}_2 &= \frac{1}{\rho A R}\sqrt{\rho g(h_1 - h_3)} - \frac{1}{\rho A R}\sqrt{\rho g h_2} \end{aligned}$$

With the same definitions for the perturbed quantities as for part (b), plus $h_3 = 20$ cm,.we obtain

$$\begin{aligned} \delta\dot{h}_1 &= -\frac{1}{(1)(100)(30)}\sqrt{(1)(1000)(30 + \delta h_1 - 20)} + \frac{1}{(1)(100)} W_{in} \\ \delta\dot{h}_2 &= \frac{1}{(1)(100)(30)}\sqrt{(1)(1000)(30 + \delta h_1 - 20)} \\ &\quad - \frac{1}{(1)(100)(30)}\sqrt{(1)(1000)(10 + \delta h_2)} \end{aligned}$$

which, with the linearization carried out, reduces to

$$\begin{aligned} \delta\dot{h}_1 &= -\frac{1}{(30)}(1 + \frac{1}{20}\delta h_1) + \frac{1}{(100)} W_{nom} + \frac{1}{(100)}\delta W \\ \delta\dot{h}_2 &= \frac{1}{(30)}(1 + \frac{1}{20}\delta h_1) - \frac{1}{(30)}(1 + \frac{1}{20}\delta h_2) \end{aligned}$$

and with the nominal flow rate of $W_{in} = \frac{10}{3}$ removed

$$\begin{aligned} \delta\dot{h}_1 &= -\frac{1}{600}\delta h_1 + \frac{1}{100}\delta W \\ \delta\dot{h}_2 &= \frac{1}{600}\delta h_1 - \frac{1}{600}\delta h_2 \end{aligned}$$

Taking the Laplace transform of these two equations, and solving for the desired transfer function (in cc/sec) yields

$$\frac{\delta H_2(s)}{\delta W(s)} = \frac{1}{600}\frac{0.01}{(s + 1/600)^2}.$$

which becomes, with the inflow in grams/min,

$$\frac{\delta H_2(s)}{\delta W(s)} = \frac{1}{600}\frac{(0.01)(60)}{(s + 1/600)^2} = .\frac{0.001}{(s + 1/600)^2}$$

27. The equations for heating a house are given by Eqs. (2.81) and (2.82) and, in a particular case can be written with time in *hours* as

$$C\frac{dT_h}{dt} = Ku - \frac{T_h - T_o}{R}$$

where

(a) C is the Thermal capacity of the house, $BTU/^oF$

(b) T_h is the temperature in the house, ^oF

(c) T_o is the temperature outside the house, ^oF

(d) K is the heat rating of the furnace, $= 90,000\ BTU/hour$

(e) R is the thermal resistance, ^oF per $BTU/hour$

(f) u is the furnace switch, =1 if the furnace is on and =0 if the furnace is off.

It is measured that, with the outside temperature at 32 ^oF and the house at 60 ^oF, the furnace raises the temperature 2 ^oF in 6 minutes (0.1 hour). With the furnace off, the house temperature falls 2 ^oF in 40 minutes. What are the values of C and R for the house?

Solution:

For the first case, the furnace is on which means $u = 1$.

$$\begin{aligned} C\frac{dT_h}{dt} &= K - \frac{1}{R}(T_h - T_o) \\ \dot{T}_h &= \frac{K}{C} - \frac{1}{RC}(T_h - T_o) \end{aligned}$$

and with the furnace off,

$$\dot{T}_h = -\frac{1}{RC}(T_h - T_o)$$

In both cases, it is a first order system and thus the solutions involve exponentials in time. The approximate answer can be obtained by simply looking at the slope of the exponential at the outset. This will be fairly accurate because the temperature is only changing by 2 degrees and this represents a small fraction of the 30 degree temperature difference. Let's solve the equation for the furnace off first

$$\frac{\Delta T_h}{\Delta t} = -\frac{1}{RC}(T_h - T_o)$$

plugging in the numbers available, the temperature falls 2 degrees in 2/3 hr, we have

$$-\frac{2}{2/3} = -\frac{1}{RC}(60 - 32)$$

which means that

$$RC = 28/3$$

For the second case, the furnace is turned on which means

$$\frac{\Delta T_h}{\Delta t} = \frac{K}{C} - \frac{1}{RC}(T_h - T_o)$$

and plugging in the numbers yields

$$\frac{2}{0.1} = \frac{90,000}{C} - \frac{1}{28/3}(60 - 32)$$

and we have

$$\begin{aligned} C &= \frac{90,000}{23} = 3913 \\ R &= \frac{RC}{C} = \frac{28/3}{3913} = 0.00239 \end{aligned}$$