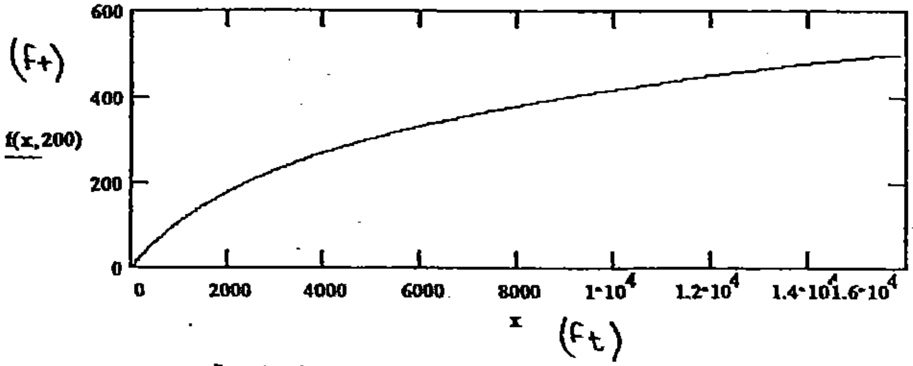
CHAPTER 2

PROBLEM 2.1

(a) $f(x) = 200 \ln\left[7.06\times10^{-4}(x+1416)\right]$ ft.

Road Profile

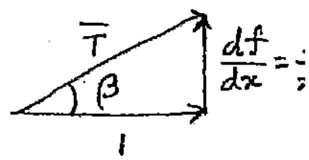



(b) The position vector is:

$$\bar{r}(x) = x\,\bar{i} + 200\ln\left[7.06\times10^{-4}(x+1416)\right]\bar{j}$$

The roadway tangent vector is:

$$\bar{T}(x) = \frac{d\bar{r}(x)}{dx} = \bar{i} + \frac{200}{x+1416}\,\bar{j}$$




$$\therefore \tan\beta = \frac{200}{x+1416}$$

$$\Rightarrow \boxed{\beta(x) = \tan^{-1}\left(\frac{200}{x+1416}\right)}$$

$$\beta(500\,ft) = \tan^{-1}\frac{200}{1916} = 5.9^\circ$$

$$\Rightarrow \boxed{\beta(500\,ft) = 5.9^\circ}$$

(c) $\%\ \text{grade} = 100\tan\beta = 100 * \frac{200}{x+1416}$

$$\therefore \boxed{\%\ \text{grade} = \frac{20{,}000}{x+1416}}$$

$$\boxed{\%\ \text{grade}\,(500\,ft) = 10.4\ \%}$$

(d) $$s(x) = \int_{\tau=0}^{x} \|\bar{T}(\tau)\|\, d\tau$$

where, $$\|\bar{T}(\tau)\| = \sqrt{1+\left(\frac{200}{\tau+1416}\right)^2}$$

Let $u = \tau + 1416$

$\Rightarrow du = d\tau$

Then, $$s(x) = \int_{u=1416}^{u=x+1416} \sqrt{1+\frac{200^2}{u^2}}\, du$$

$$= \int_{1416}^{x+1416} \frac{\sqrt{u^2+200^2}}{u}\, du$$

$$\Rightarrow s(x) = \left\{ \sqrt{u^2+200^2} - 200 \ln\left[\frac{200+\sqrt{u^2+200^2}}{u}\right]\right\}\Bigg|_{1416}^{x+1416}$$

[FROM TABLE OF INTEGRALS]

$$\Rightarrow s(x) = \sqrt{(x+1416)^2+200^2} - 200 \ln\left[\frac{200+\sqrt{(x+1416)^2+200^2}}{x+1416}\right] - \sqrt{1416^2+200^2} + 200 \ln\left[\frac{200+\sqrt{1416^2+200^2}}{1416}\right]$$

$$\Rightarrow \boxed{s(x) = \sqrt{(x+1416)^2+200^2} - 200 \ln\left[\frac{200+\sqrt{(x+1416)^2+200^2}}{x+1416}\right] - 1402}$$

$$\boxed{s(500\text{ft.}) = 507\text{ ft.}}$$

## PROBLEM 2.2

$$\bar{r}(x) = x_f\,\bar{i}_f + 4.1\sqrt{x_f}\,\bar{j}_f$$

$$\bar{T}(x) = \frac{d\bar{r}}{dx} = \bar{i}_f + \frac{df}{dx}\,\bar{j}_f$$

(a) $$\beta(x_f) = \tan^{-1}\frac{df}{dx_f} = \tan^{-1}\frac{2.05}{\sqrt{x_f}}$$

$$\beta(x_f)\Big|_{x_f = 5280} = \tan^{-1}\frac{2.05}{\sqrt{5280}} = 1.62^\circ$$

(b) $$\|T\| = \sqrt{1+\left(\frac{df}{dx_f}\right)^2}$$

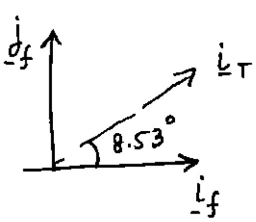
$$s = \int_0^{2\,mi}\|T\|\,dx_f = \int_0^{5280}\sqrt{1+\left(\frac{2.05}{\sqrt{x_f}}\right)^2}\,dx_f$$

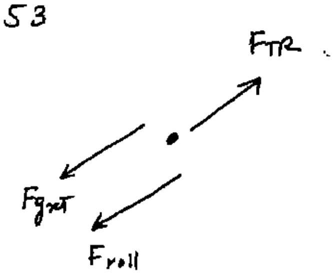
THE INTEGRAL CAN BE SOLVED NUMERICALLY.

$\Rightarrow$ $S = 5300\ ft.$

## PROBLEM 2.3

(a) $$\beta = \tan^{-1}\left(\frac{15}{100}\right) = 8.53^\circ$$




EV at rest

$\Rightarrow$ $\sum F = 0$

$\Rightarrow$ $F_{TR} - F_{gxT} - F_{roll} = 0$

(i) if $F_{roll} = 0$ $\Rightarrow$ $F_{TR} = F_{gxT}$

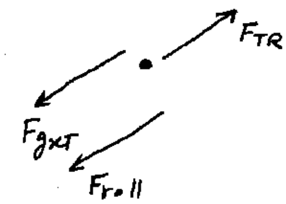
$$= mg\sin\beta = 692 * 9.81 * \sin 8.53$$

$$= 1007\ N$$

$\Rightarrow$ $\boxed{F_{TR} = 1007N}$

(ii)

Since the vehicle is at standstill,




$$F_{roll} = \left(F_{TR} - F_{gxT}\right) \quad \text{if} \quad v_{xT} = 0 \quad \text{and} \quad \left|F_{TR} - F_{gxT}\right| \leq c_0 m$$

$F_{roll}$ opposes the backward motion of the vehicle. Therefore, the maximum rolling resistance in this case is,

$$\left|F_{roll}\right| = c_0 mg$$

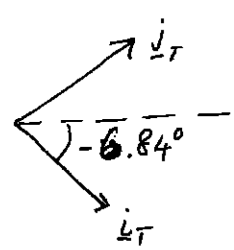
and $\quad F_{roll} = -c_0 mg$

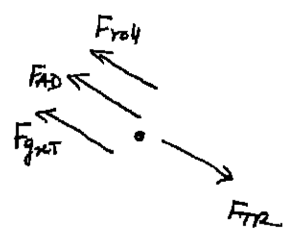
Therefore, $\quad F_{TR}\big|_{min} - F_{gxT} = -c_0 mg$

$$\Rightarrow F_{TR}\big|_{min} = F_{gxT} - c_0 mg = 1007 - (0.009)(6789)$$
$$= 946\ N$$

$F_{TR}\big|_{min}$ is the minimum tractive force required to prevent the vehicle from rolling backwards.

(b) $\quad \beta = \tan^{-1}\left(\frac{-12}{100}\right) = -6.84^0$




$$F_{TR} = F_{gxT} + F_{AD} + F_{roll}$$

$$F_{gxT} = mg \sin\beta$$
$$= 692 \times 9.81 \sin(-6.84)$$
$$= -809\ N$$

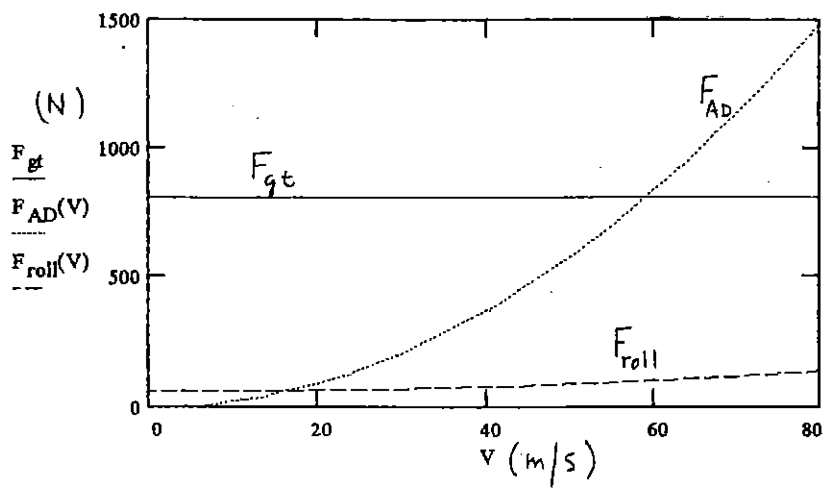
$$F_{AD} = \frac{\rho}{2} c_D A_F v^2$$
$$= .232\, v^2$$

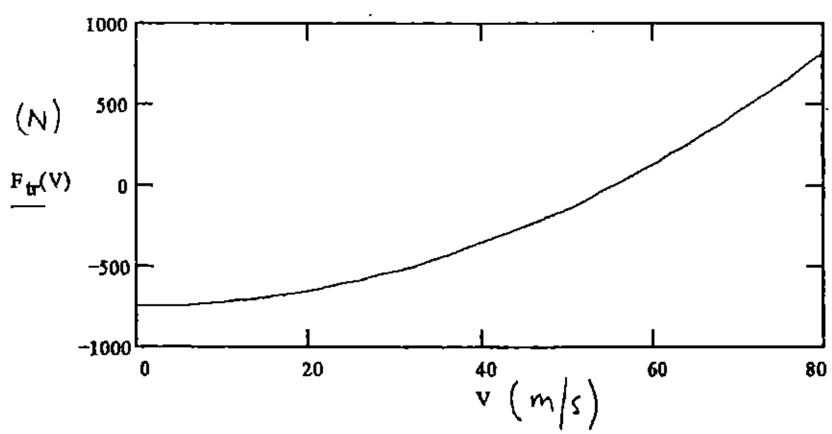
$$F_{roll} = mg\left(c_0 + c_1 v^2\right)$$
$$= 6789\left(.009 + 1.75 \times 10^{-6} v^2\right)$$
$$= 61.1 + .012 v^2$$

$$\Rightarrow F_{TR} = 0.232v^2 + 61.1 + .012v^2 - 809$$

$$\Rightarrow \boxed{F_{TR} = 0.244v^2 - 748} \text{ N}$$

PLOTS



PROBLEM 2.4

For $t > 0$

$$\Sigma F = m\frac{dv}{dt}$$

$$\Rightarrow F_{TR} - \operatorname{sgn}[v]\left[\frac{\rho}{2} C_D A_F\right] v^2 - \operatorname{sgn}(v)\, mg\,(C_0 + C_1 v^2) - mg\sin\beta = m\frac{dv}{dt}$$

$$\Rightarrow \frac{dv}{dt} = \left[\frac{F_{TR}}{m} - g\,(\sin\beta + C_0)\right] - \left[\frac{\rho}{2} C_D A_F + g C_1\right] v^2$$

$[\operatorname{sgn}[v] = 1$ since $v(t) \geq 0$

Let $K_7 = \frac{F_{TR}}{m} - g\,(\sin\beta + C_0)$

$K_8 = \frac{\rho}{2m} C_D A_F + g C_1$

note, $K_8 > 0$. Also $K_7 > 0$ since it is required that $\frac{dv(0)}{dt} > 0$ in order to satisfy $v(t) \geq 0$ for $t \geq 0$

And $\frac{dv(0)}{dt} > 0 \Rightarrow F_{gxT} + F_{TR} > C_0\, mg$.

$$\frac{dv}{dt} = K_7 - K_8 v^2$$

$$\Rightarrow \frac{dv}{\frac{K_7}{K_8} - v^2} = K_2\, dt \quad \Rightarrow \int_{v(0)=0}^{v(t)} \frac{dv}{\left(\sqrt{\frac{K_7}{K_8}}\right)^2 - v^2} = K_8\int_0^t dt$$

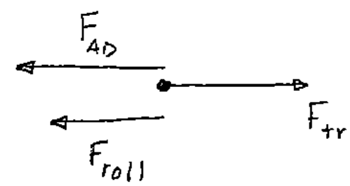
$$\Rightarrow \frac{1}{\sqrt{\frac{K_7}{K_8}}} \tanh^{-1}\left(\sqrt{\frac{K_8}{K_7}}\, v\right) = K_8\, t$$

$$\Rightarrow v(t) = V_T \tanh\left(\sqrt{K_7 K_8}\; t\right)$$

where $V_T = \sqrt{\frac{K_7}{K_8}}$

## PROBLEM 2.5

(a)




$$F_{tr} - F_{AD} - F_{roll} = m\frac{dv}{dt}$$

$$F_{tr} = m\frac{dv}{dt} + F_{AD} + F_{roll}$$

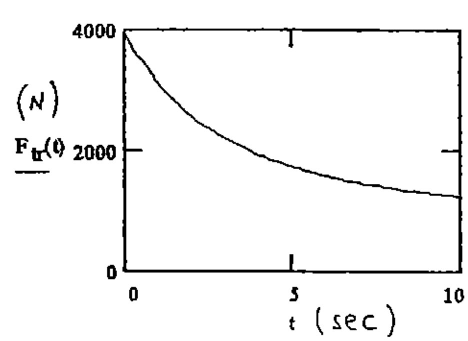
$$F_{tr}(t) = \frac{692(20)(0.282)}{0.282\,t+1} + \frac{1.16}{2}(0.2)(2)\,v^2 + (692)(9.81)\left[0.009 + 1.75\cdot 10^{-6} v\right.$$

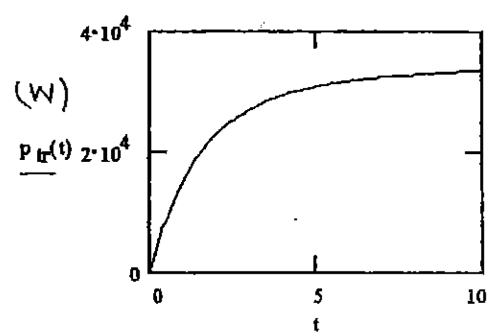
$$F_{tr}(t) = \frac{3903}{0.282\,t+1} + 61.1 + 0.244\,v^2$$

$$F_{tr}(t) = \frac{3903}{0.282\,t+1} + 61.1 + 97.6\left[\ln(0.282\,t+1)\right]^2$$

$$P_{tr}(t) = F_{tr}(t)\,v(t)$$

$$= \frac{78060}{0.282\,t+1}\ln(0.282\,t+1) + 1222\ln(0.282\,t+1) + 1952\left[\ln(0.282+1)\right]^3$$



$$1951 \int_0^{10} \left[\ln(0.282t+1)\right]^3 dt = \frac{1951}{0.282} \int_1^{3.82} (\ln u)^3 \, du$$

let $u = 0.282t + 1$

$du = 0.282\,dt$

$$= 6918 \left\{ u(\ln u)^3 - 3\left[u(\ln u)^2 - 2u\ln u + 2u\right]\right\}\Bigg|_1^{3.82}$$

$$= 6918 \{ 9.20 - 3[6.86 - 10.2 + 7.64] - 0 + 3[0-0+2$$

$$= 6918(2.3) = 15911 \text{ J}$$

$$\int_0^{10} v\left[F_{AD} + F_{roll}\right] dt = 9965 + 15911 = 25876 \text{ J}$$

so $\Delta \ell_{tr} = 2.485 \cdot 10^5 + 2.5876 \cdot 10^4$

$$\boxed{\Delta \ell_{tr} = 2.744 \cdot 10^5 \text{ J}}$$

$$\frac{\Delta KE}{\Delta \ell_{tr}} \cdot 100\% = 90.6\%$$

$$\frac{\int_0^{10} v\left[F_{AD} + F_{roll}\right] dt}{\Delta \ell_{tr}} \cdot 100\% = 9.4\%$$

(b) $\Delta \ell_{tr} = \Delta KE + \Delta PE + \int_0^{10 sec} v(F_{AD} + F_{roll})\,dt$

$$\Delta KE = \frac{1}{2} m \left[ v^2(10 sec) - v^2(0) \right]$$

$$= \frac{1}{2}(692)(26.8)^2 = 2.485 \cdot 10^5 \text{ J}$$

$\Delta PE = 0$ since road is level

$$\int_0^{10 sec} v(F_{AD} + F_{roll})\,dt = \int_0^{10 sec} (692)(9.81)(0.009)\,20 \ln(0.282t+1)\,dt$$

$$+ \int_0^{10} \left[ 692(9.81)(1.75 \cdot 10^{-6}) + \frac{1.16}{2}(0.2)(2) \right] \left[ 20 \ln(0.282t+1) \right] dt$$

$$= 1222 \int_0^{10} \ln(0.282t+1)\,dt + 1951 \int_0^{10} \left[ \ln(0.282t+1) \right]^3 dt$$

let $u = 0.282t + 1$

$du = 0.282\,dt$

$t = 0 \Rightarrow u = 1$

$t = 10 \Rightarrow u = 3.82$

$$1222 \int_0^{10} \ln(0.282t+1)\,dt = \frac{1222}{0.282} \int_1^{3.82} \ln u\,du$$

$$= 4333 \left[ u \ln u - u \right]_1^{3.82}$$

$$= 4333 \left[ 3.82 \ln 3.82 - 3.82 - \cancel{\ln 1}^{\,0} + 1 \right] = 9965 \text{ J}$$

$$F_{tr} = \left[ \operatorname{sgn}(V)\, mg\, C_0 + mg \sin\beta \right] + \operatorname{sgn}(V) \left[ \frac{\rho}{2} C_D A_f + mg\, C_1 \right] V^2$$

$$F_{tr} = \left[ \operatorname{sgn}(V)(692)(9.81)(0.009) + (692)(9.81)\sin\beta \right]$$
$$+ \operatorname{sgn}(V) \left[ \frac{1.16}{2}(0.2)(2) + (692)(9.81)(1.75 \cdot 10^{-6}) \right] \cdot V^2$$

$$F_{tr} = 61.1 \operatorname{sgn}(V) + 6789 \sin\beta + 0.244 V^2 \operatorname{sgn}(V)$$

if $V > 0$ then

$$F_{tr} = 61.1 + 6789 \sin\beta + 0.244 V^2$$

$\beta = 0° \Rightarrow F_{tr} = 61.1 + 0.244 V^2$

$\beta = 4° \Rightarrow F_{tr} = 535 + 0.244 V^2$

$\beta = -4° \Rightarrow F_{tr} = -412 + 0.244 V^2$

if $V < 0$ then

$$F_{tr} = -61.1 + 6789 \sin\beta - 0.244 V^2$$

$\beta = 0° \Rightarrow F_{tr} = -61.1 - 0.244 V^2$

$\beta = 4° \Rightarrow F_{tr} = 412 - 0.244 V^2$

$\beta = -4° \Rightarrow F_{tr} = -535 - 0.244 V^2$

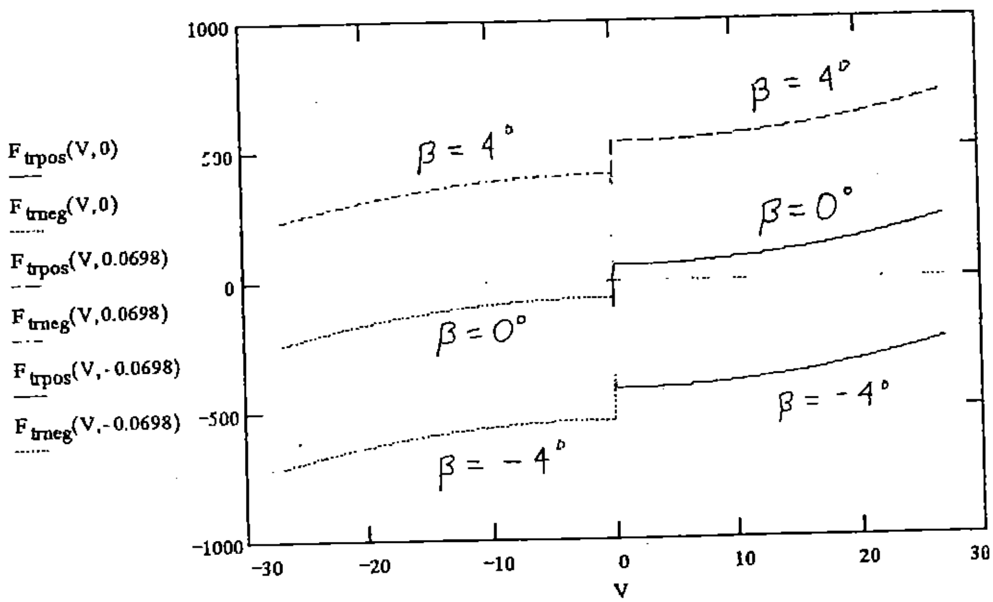
PLOTS FOR 3 STEADY STATE VELOCITIES

$V := -27, -26.9 .. 27$

$F_{trpos}(V, \beta) := \left(61.1 + 6789 \cdot \sin(\beta) + 0.244 \cdot V^2\right) \cdot \Phi(V)$

$F_{trneg}(V, \beta) := \left(-61.1 + 6789 \cdot \sin(\beta) - 0.244 \cdot V^2\right) \cdot \Phi(-V)$

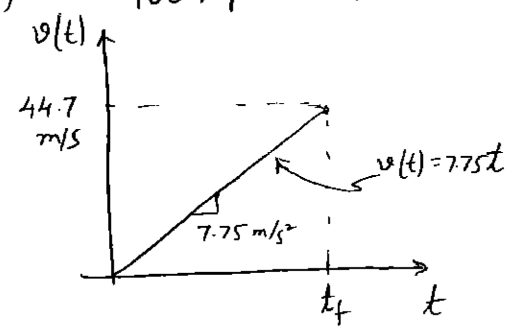
Steady-State Tractive Force versus Constant Vehicle Velocity
(beta = 0, +4, -4 degrees)




Units: Vert. axis Newtons, Hor. axis m/sec.

## PROBLEM 2.7

(a) 100 mph = 44.7 m/s




$$\int_0^{t_f} v\,dt = 100\,m + \frac{25\,m}{\cos 30^\circ}$$

$$\Rightarrow \frac{1}{2} t_f * 44.7 = 100\,m + 28.9$$

$$\Rightarrow t_f = 5.77 \text{ secs.}$$

(b) $\Delta P.E. = mg\left[ f(t_f) - f(t_i) \right] = 692 \times 9.81 \times \underbrace{(25 \tan 30^\circ)}_{14.43}$

$= 9.8 \times 10^4$ Joules.

(c) $E_{Loss} = \left[ mg\, c_0 \overbrace{\textstyle\int_0^{t_f} v\,dt} + \left(mg\, c_1 + \frac{\rho}{2} C_D A_F\right) \int_0^{t_f} v^3\,dt \right] = 3.93 \times 10^4$ J

ALTERNATIVELY,

$\Delta e_{TR} = \Delta K.E. + \Delta P.E. + E_{Loss}$

$\Rightarrow E_{Loss} = \Delta e_{TR} - \Delta P.E. - \Delta K.E.$

$= 8.28 \times 10^5 - .98 \times 10^5 - 6.91 \times 10^5 = 3.9 \times 10^4$ J

$\Delta K.E. = \frac{1}{2}\, 693 * 44.7^2$

$= 6.91 \times 10^5$

**Solution to Problem # 2.8**

The vehicle parameters and performance requirements of the vehicle are used for sizing calculations:

| **Description** | **Requirements** |
|---|---|
| Vehicle Mass | 2000kg |
| Driver/One passenger | 176 Lbs./ 80 kg |
| Rolling Resistance Coefficient, | $C_0 = 0.01$; |
| Wheel Radius, $r_{wh}$ | 0.3305 m |
| Aerodynamic Drag Coefficient, $C_{AD}$ | 0.45 |
| Frontal Area, $A_F$ | 2.5m$^2$ |

Mass of driver and 1 passenger,

$m = 2000 + 2 \times 80 = 2160kg$

For a grade of 0.5%, $\beta = \tan^{-1}(.5/100) = 0.2865^0$

(a) *With constant force $P_{TR}$ acceleration*

$$\frac{P_{TR}}{v} = m\frac{dv}{dt} + mg[\sin\beta + C_0] + \left[mgC_1 + \frac{\rho}{2}C_D A_F\right]v^2$$

$$\Rightarrow \frac{dv}{dt} = \frac{P_{TR}}{mv} + g[\sin\beta + C_0] + \left[gC_1 + \frac{\rho}{2m}C_D A_F\right]v^2 \qquad \text{(P2.81)}$$

(b) Using a Matlab program with equation (1) and given initial conditions at 5 s, and then continuing with constant ($P_{TR}$ =145kW), the following data s obtained

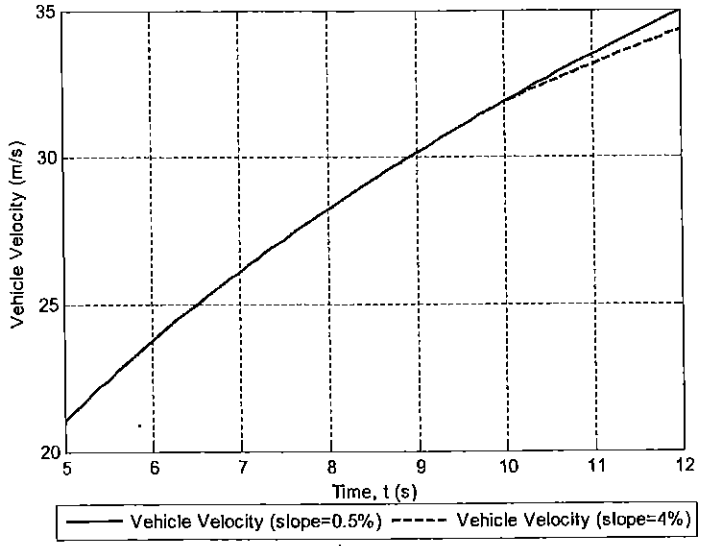
At 10 secs., velocity is 31.99 m/s (= 71.57 mi/h).

The Matlab program used is included at the end of this problem solution

(c ) At 12 s, with a slope change to 4% at t=10 s, the vehicle velocity at t = 12 s is 33.14 m/s ( =74.14 mi/h)

If the slope remained at 0.5%, then the vehicle velocity at t = 12 s is 34.9686 m/s ( =78.23 mi/h)

The plots for both the scenarios is given below:




## Matlab program for constant power operation

```
% Problem 2.8 (b) and (c) of book Solution
clear;
echo off;

Co=.01;
Cd=0.45;
Af=2.5;
g=9.81;
m=2000+160;

row=1.115;
grade=0.5; % 0.5% grade
angle=atan(0.5/100);


% Vehicle accelerates from 0 to 21 m/s in 5 s on a 0.5% grade. Then it
accelerates with a
%  constant power of 145kW
v_hev(1)=21;% vehicle velocity
b(1)=5; %Time

PTR_const=145000;

PTR(1)=PTR_const;


    kkmax=121-50; % Variable for simulation up to 12 s
    deltat= 0.1;  % step time
for kk=2:kkmax
    t=(kk-1)*deltat;
    b(kk)=t+5;
```

```
        v_hev(kk)=v_hev(kk-1)+(PTR_const/(m*v_hev(kk-1))-g*(sin(angle)+Co)-
(0.5*row*Af*Cd*v_hev(kk-1)^2)/m)*deltat;
        FTR(kk)=PTR_const/v_hev(kk); % Traction force
        PTR(kk)=FTR(kk)*v_hev(kk);  % Traction Power (constant at 145 kW)

end

v_hev1(1)=21;% vehicle velocity
b1(1)=5; %Time

for kk=2:kkmax
    t=(kk-1)*deltat;
    b1(kk)=t+5;
     if b1(kk)>10
         angle=atan(4/100); %grade changes to 4% at t=10 s
     else
     end
        v_hev1(kk)=v_hev1(kk-1)+(PTR_const/(m*v_hev1(kk-1))-g*(sin(angle)+Co)-
(0.5*row*Af*Cd*v_hev1(kk-1)^2)/m)*deltat;

end

% Plotting speed vs time
plot(b,v_hev,'-',b1,v_hev1,'--','Linewidth',1.5);
grid;

xlabel('Time, t (s)');
ylabel('Vehicle Velocity (m/s)');

legend('Vehicle Velocity (slope=0.5%)','Vehicle Velocity
(slope=4%)','Location','SouthOutside','Orientation','horizontal');
clc;
```

**Solution to Problem # 2.9**

(a)

The slip s of a vehicle is given

$s = (1 - \dfrac{\varsigma_T v}{\omega_{wh} r_{wh}}) \times 100\,\%$ where $\zeta_T$ is the total gear ratio from engine/motor to driven wheels, $\omega_{wh}$ is the wheel speed and $v$ is the vehicle velocity.

The vehicle velocity is then given by

$$v = \frac{\omega_{wh} r_{wh}}{\varsigma_T}(1-s) \quad m/s$$

The wheel speed is then $\Rightarrow \omega_{wh} = \dfrac{26.8*1}{0.3305(1-.15)} = 95.4 \quad rads/\sec$ (assuming $\zeta_T$=1).

(b)

The traction force limit is

$$\mu_{tr\_limit} = \mu(s) * F_{gyT}$$

$F_{gyT}$ is the normal force and is equal to $mg\cos\beta = 1960*9.81*\cos(.57) = 19{,}227.6N$

(For a grade of 1%, $\beta = \tan^{-1}(1/100) = 0.57^0$)

$$\mu(15\%) = \mu_{pk}\left[a(1-e^{-bs}) - cs\right] = 0.85*\left[1.1(1-e^{-20*.15}) - .0035*.15\right] = 0.85*(1.0452 - .0005) = 0.888$$

for $\mu_{pk} = 0.85, \quad a = 1.1, \quad b = 20, \quad and \quad c = .0035$

Therefore,

$$\mu_{tr\_limit} = \mu(15\%)*mg\cos\beta = 0.888*19{,}227.6 = 17{,}074N$$