Chapter 3

Test Bank

3.0 Chapter 0

1. Convert the binary number 10110 to base ten.

   Answer. 22

2. Write the base ten number 37 in binary.

   Answer. 100101

3. Write the base ten number 19 in binary.

   Answer. 10011

4. Convert the base 8 number 75 to base ten.

   Answer. 61

5. Write the base ten number 75 in base 8.

   Answer. 113

6. Convert the base 16 number $a7$ to base ten.

   Answer. 167

7. Write the base ten number 436 in base 16.

   Answer. $1b4$

8. How is 8^n expressed in binary?

   Answer. A 1 followed by $3n$ 0's.

3.1 Chapter 1

Section 1.1

1. Is the sentence "There are no true sentences." a statement? Explain.

   Answer. Yes. It is false.

2. Make a truth table for $p \to q \vee r$.

   Answer.

p	q	r	$q \vee r$	$p \to q \vee r$
F	F	F	F	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T
T	F	F	F	F
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

3. Is the statement form $p \to \neg p$ a contradiction? Explain.

   Answer. No. It is true when p is false.

4. Determine if $\neg(p \to q)$ and $\neg p \to \neg q$ are logically equivalent? Justify your answer.

   Answer. They are not logically equivalent. They differ when p is true and q is true.

5. Write and simplify the contrapositive of $p \to \neg q \wedge r$.

   Answer. $q \vee \neg r \to \neg p$

6. Given the statement

   If Tara is not studying, then Tara is sleeping.

   Write its

   (a) converse.

   (b) contrapositive.

   (c) inverse.

   (d) negation.

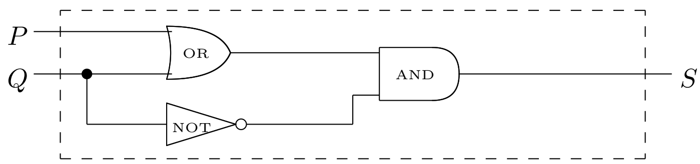
Answer.
(a) If Tara is sleeping, then Tara is not studying.
(b) If Tara is not sleeping, then Tara is studying.
(c) If Tara is studying, then Tara is not sleeping.
(d) Tara is not studying, and Tara is not sleeping.

7. Verify that $\neg p \wedge (\neg q \vee p) \equiv \neg(p \vee q)$ not by making a truth table but by using known basic logical equivalences.

Answer.

$$\begin{array}{rcll} \neg p \wedge (\neg q \vee p) & \equiv & (\neg p \wedge \neg q) \vee (\neg p \wedge p) & \text{Distributivity} \\ & \equiv & (\neg p \wedge \neg q) \vee \underline{f} & \text{Contradiction Rule} \\ & \equiv & \neg p \wedge \neg q & \text{Contradiction Rule} \\ & \equiv & \neg(p \vee q) & \text{De Morgan's Law} \end{array}$$

8. Trace the pictured circuit

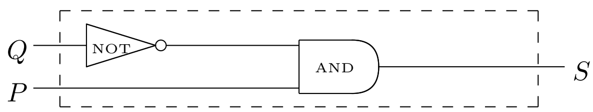


(a) to determine an expression for the output in terms of the input,
(b) and make an input-output table.
(c) Explain how the same input-output table can be accomplished by a circuit using fewer basic gates.

Answer.
(a) $(P \vee Q) \wedge \neg Q = S$.
(b)

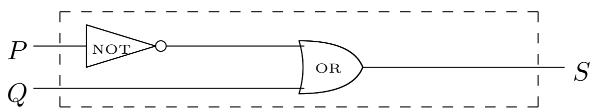
P	Q	S
0	0	0
0	1	0
1	0	1
1	1	0

(c) $S \equiv P \wedge \neg Q$.



9. Draw a circuit that realizes the expression $\neg P \vee Q = S$.

Answer.



Section 1.2

1. Express in set notation the set of integers smaller than 5.

   Answer. $\{n : n \in \mathbb{Z} \text{ and } n < 5\}$.

2. Express in interval notation the set of real numbers greater than or equal to -3.

   Answer. $[-3, \infty)$.

   For Exercises 3 through 8, determine if each of the the following relations is True or False.

3. $\{1, 3, 5, 3, 1, 7, 1\} \subseteq \{1, 3, 5, 7\}$.

   Answer. True.

4. $\{7\} \in \mathbb{N}$.

   Answer. False.

5. $3 \subset \{1, 2, 3, 4\}$.

   Answer. False.

6. $\emptyset = 0$.

   Answer. False.

7. $[-1, 1]$ is infinite.

   Answer. True.

8. $|\{2, 3, 7, 8, 5, 3\}| = 6$.

   Answer. False.

9. Write the expression for the "set" given in Russell's Paradox.

   Answer. $\{S : S \text{ is a set and } S \notin S\}$.

Section 1.3

For Exercises 1 through 3, write the given statement as efficiently as possible using quantifiers and standard notation. Determine if the statement is True or False.

1. Every real number is smaller than twice itself.

   Answer. $\forall\, x \in \mathbb{R}, x < 2x$.

2. There is an integer whose square is odd.

Answer. $\exists\, n \in \mathbb{Z}$ such that n^2 is odd.

3. There is an integer n such that the n^{th} power of every real number is negative.

   Answer. $\exists\, n \in \mathbb{Z}$ such that $\forall\, x \in \mathbb{R}, x^n < 0$.

For Exercises 4 through 6, write the negation of the given statement. Determine which of the statement or its negation is True.

4. For every integer n, if n is positive then $2n - 1$ is positive.

   Answer. $\exists\, n \in \mathbb{Z}$ such that $n > 0$ and $2n - 1 \leq 0$.
   The original statement is True.

5. There is a real number whose cube is negative.

   Answer. $\forall\, x \in \mathbb{R}, x^3 \geq 0$.
   The original statement is True.

6. The product of any two real numbers is positive.

   Answer. $\exists\, x, y \in \mathbb{R}$ such that $xy \leq 0$.
   The negation is True.

7. Negate the statement

$$\exists\, n \in \mathbb{Z} \text{ such that } \forall\, x \in \mathbb{R}, x^n < 0.$$

   Answer. $\forall\, n \in \mathbb{Z}, \exists\, x \in \mathbb{R}$ such that $x^n \geq 0$.

8. Negate the statement

   All good things come to an end.

   Answer. There is a good thing that does not end.

For Exercises 9 and 10, let f and g be real functions. Use quantifiers to precisely express the definition of the given notion.

9. f is periodic.

   Answer. $\exists\, p \in \mathbb{R}^+$ such that $\forall\, x \in \mathbb{R}, f(x + p) = f(x)$.

10. The composite function $g \circ f$.

    Answer. The function $g \circ f$ is defined by

$$\forall\, x \in \mathbb{R}, (g \circ f)(x) = g(f(x)).$$

Section 1.4

For Exercises 1 and 2, find A^c, $A \cap B$, $A \cup B$, $A \setminus B$, and $A \triangle B$ for the given sets.

1. $A = \{2, 3, 7\}$, $B = \{1, 2, 7, 9\}$, and $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

   Answer. $A^c = \{1, 4, 5, 6, 8, 9, 10\}$, $A \cap B = \{2, 7\}$, $A \cup B = \{1, 2, 3, 7, 9\}$, $A \setminus B = \{3\}$, and $A \triangle B = \{1, 3, 9\}$.

2. $A = (0, 3]$, $B = (2, 4)$, and $\mathcal{U} = \mathbb{R}$.

   Answer. $A^c = (-\infty, 0] \cup (3, \infty)$, $A \cap B = (2, 3]$, $A \cup B = (0, 4)$, $A \setminus B = (0, 2]$, and $A \triangle B = (0, 2] \cup (3, 4)$.

3. Are $(0, 3)$ and $(2, 4)$ disjoint? Justify your answer.

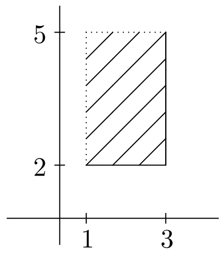
   Answer. No. $2.5 \in (0, 3) \cap (2, 4) \neq \emptyset$.

4. Find $\{0, 1\} \times \{2, 4, 6\}$.

   Answer. $\{(0, 2), (0, 4), (0, 6), (1, 2), (1, 4), (1, 6)\}$.

5. Sketch $(1, 3] \times [2, 5)$.

   Answer.



6. Find $\mathcal{P}(\{0, 1, 2\})$.

   Answer. $\{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.

7. Decide if the proposed identity $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$ is True or False.

   Answer. True.

8. Use definitions and basic set identities to verify the identity
$$A^c \cap (B^c \cup A) = (A \cup B)^c.$$

   Answer.

$A^c \cap (B^c \cup A)$	$=$	$(A^c \cap B^c) \cup (A^c \cap A)$	Distributivity
	$=$	$(A^c \cap B^c) \cup \emptyset$	An $\emptyset$ Rule
	$=$	$A^c \cap B^c$	An $\emptyset$ Rule
	$=$	$(A \cup B)^c$	De Morgan's Law

Section 1.5

1. Determine if the given argument form is valid. Justify your answer.

$$\begin{array}{l} p \to q \\ r \to p \\ q \lor r \\ \therefore \; q \end{array}$$

Answer.

p	q	r	$p \to q$	$r \to p$	$q \lor r$	q
F	F	F	T	T	F	
F	F	T	T	F	T	
F	T	F	T	T	T	T
F	T	T	T	F	T	
T	F	F	F	T	F	
T	F	T	F	T	T	
T	T	F	T	T	T	T
T	T	T	T	T	T	T

Rows 3, 7, and 8 demonstrate the validity of the argument form.

2. Show that the given argument form is valid without using a truth table.

$$\begin{array}{l} q \to p \\ \neg q \to p \\ \therefore \; p \end{array}$$

Answer.

	Statement Form	Justification
1.	$q \to p$	Given
2.	$\neg q \to p$	Given
3.	$q \lor \neg q$	a tautology
4.	$\therefore \; p$	(1),(2),(3), Two Separate Cases

3. Determine if the given argument is valid or invalid. Justify your answer.

$$\begin{array}{l} \text{If } e > 0, \text{ then } \dfrac{1}{e} > 0. \\ \dfrac{1}{e} > 0. \\ \therefore \; e > 0. \end{array}$$

Answer. The argument's form

$$\begin{array}{l} p \to q \\ q \\ \therefore \; p \end{array}$$

is not valid, as can be seen when p is false (and q is arbitrary). So the argument is not valid.

4. Verify that the given argument form is valid.

$$\begin{array}{l} \forall\, x \in \mathcal{U}, p(x) \wedge q(x) \\ a \in \mathcal{U} \\ \therefore\ \ p(a) \end{array}$$

Answer.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x) \wedge q(x)$	Given
2.	$a \in \mathcal{U}$	Given
3.	$p(a) \wedge q(a)$	(1),(2), Principle of Specification
4.	$\therefore\ \ p(a)$	(3), In Particular

5. Verify that the given argument form is valid.

$$\begin{array}{l} \forall\, x \in \mathcal{U}, p(x) \\ \therefore\ \ \forall\, x \in \mathcal{U}, p(x) \vee q(x) \end{array}$$

Answer.

	Statement Form	Justification
1.	$\forall\, x \in \mathcal{U}, p(x)$	Given
2.	Let $a \in \mathcal{U}$ be arbitrary	Assumption
3.	$p(a)$	(1),(2), Principle of Specification
4.	$p(a) \vee q(a)$	(3), Obtaining Or
5.	$\therefore\ \ \forall\, x \in \mathcal{U}, p(x) \vee q(x)$	(2), (4), Principle of Generalization

6. Show that the given argument form is invalid.

$$\begin{array}{l} \forall\, x \in \mathcal{U}, p(x) \to q(x) \\ \forall\, x \in \mathcal{U}, q(x) \\ \therefore\ \ \forall\, x \in \mathcal{U}, p(x) \end{array}$$

Answer. Let $\mathcal{U} = \mathbb{R}^+$, $p(x) =$ "$x > 1$", and $q(x) =$ "$x > 0$". The resulting argument

$$\begin{array}{l} \forall\, x \in \mathbb{R}^+,\ \text{if } x > 1 \text{ then } x > 0 \\ \forall\, x \in \mathbb{R}^+, x > 0 \\ \therefore\ \ \forall\, x \in \mathbb{R}^+, x > 1 \end{array}$$

has all of its premises true but its conclusion false.

3.2 Chapter 2

Section 2.1

1. Show: There exists $x \in \mathbb{Z}$ such that $2x^2 - 5x + 2 = 0$.

   Answer. Observe that $2(2^2) - 5(2) + 2 = 0$.

2. Show: There exist $m, n \in \mathbb{Z}$ such that $5m + 3n = 1$.

   Answer. Observe that $5(-1) + 3(2) = 1$.

3. Disprove: For all sets A and B, $|A \cup B| = |A| + |B|$.

   Answer. Let $A = B = \{6\}$.
   So $|A \cup B| = |\{6\}| = 1$ and $|A| + |B| = 1 + 1 = 2$.
   Hence, $|A \cup B| \neq |A| + |B|$ in this case.

4. Prove or Disprove: $\forall\, m \in \mathbb{Z}$, if m^2 is odd, then m is even.

   Answer. Counterexample: Observe that 1^2 is odd and 1 is not even.

5. Show: $\forall\, n \in \{3, 6, 9\}$, the sum of the (base ten) digits of $7n$ is n.

   Answer. Observe that
   $7(3) = 21$ and $2 + 1 = 3$,
   $7(6) = 42$ and $4 + 2 = 6$, and
   $7(9) = 63$ and $6 + 3 = 9$.

6. Show: $\forall\, A \in \mathcal{P}(\{4, 7\}), |A| \leq 2$.

   Answer. Note that $|\emptyset| = 0$, $|\{4\}| = |\{7\}| = 1$, and $|\{4, 7\}| = 2$.

Section 2.2

1. Show: $\forall\, n \in \mathbb{Z}^-, -n - 1 \in \mathbb{N}$.

   Answer. Let $n \in \mathbb{Z}^-$. So $n \in \mathbb{Z}$ and $n \leq -1$. Thus, $-n \geq 1$, and hence $-n - 1 \geq 0$. Since $-n - 1 \in \mathbb{Z}$, it follows that $-n - 1 \in \mathbb{N}$.

2. Show: $\forall\, x \in \mathbb{R}$, if $x < 0$ then $x^3 < 0$.

   Answer. Suppose $x \in \mathbb{R}$ and $x < 0$. Since $x^2 > 0$, it follows that $x(x^2) < 0(x^2)$. That is, $x^3 < 0$.

3. Show: For all real functions f, if f is bounded above, then $-2f$ is bounded below.

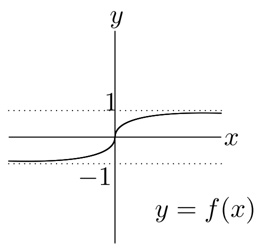
Answer. Suppose f is a real function that is bounded above. So we have $M \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R}, f(x) \leq M$. Observe that $\forall\, x \in \mathbb{R}, (-2f)(x) = -2 \cdot f(x) \geq -2 \cdot M = -2M$. Hence, $-2f$ is bounded below.

4. Show: For all real functions f, if f is decreasing, then $-f$ is increasing.

   Answer. Suppose f is a real function that is decreasing. Suppose $x, y \in \mathbb{R}$ with $x < y$. So $f(x) > f(y)$. Multiplication by -1 gives $-f(x) < -f(y)$. So $-f$ is increasing.

5. Prove or Disprove: For all real functions f, if f is increasing, then f is not bounded above.

   Answer. *Counterexample*: Let $f(x) = \frac{x}{\sqrt{x^2+1}}$.



   This function is increasing and bounded above (by 1).

6. Show: For all sets A, B, and C, $A \cap B \cap C \subseteq A \cap C$.

   Answer. Let A, B, and C be sets. Suppose that $x \in A \cap B \cap C$. So $x \in A$, $x \in B$, and $x \in C$. In particular, $x \in A$ and $x \in C$. Therefore, $x \in A \cap C$.

7. Let A and B be sets in some universal set $\mathcal{U}$. Show: $B \cup A = A \cup B$.

   Answer. Observe that $\forall\, x \in \mathcal{U}$, $x \in B \cup A$ iff $x \in B \vee x \in A$ iff $x \in A \vee x \in B$ iff $x \in A \cup B$.

Section 2.3

1. Show: $\forall\, x \in \mathbb{R}, x \in [-3, 4)$ if and only if $2x + 3 \in [-3, 11)$.

   Answer. Let $x \in \mathbb{R}$.
   $(\rightarrow)$ Suppose $x \in [-3, 4)$. That is, $-3 \leq x < 4$. So $-6 \leq 2x < 8$. So $-3 \leq 2x + 3 < 11$. That is, $2x + 3 \in [-3, 11)$.
   $(\leftarrow)$ Suppose $2x + 3 \in [-3, 11)$. That is, $-3 \leq 2x + 3 < 11$. So $-6 \leq 2x < 8$. So $-3 \leq x < 4$. That is, $x \in [-3, 4)$.

2. Let $n \in \mathbb{Z}$. Show: $2n^2 - 5n - 3 = 0$ if and only if $n = 3$.

   Answer. ($\rightarrow$) Suppose $2n^2 - 5n - 3 = 0$. So $(2n+1)(n-3) = 0$, and hence $n = -\frac{1}{2}$ or $n = 3$. Since $n \in \mathbb{Z}$, it must be that $n = 3$. ($\leftarrow$) Suppose $n = 3$. Observe that $2 \cdot 3^2 - 5 \cdot 3 - 3 = 0$.

3. Let f be a real function.
   Show: $-f$ is periodic if and only if f is periodic.

   Answer. ($\rightarrow$) Suppose $-f$ is periodic. Let p be its period. Suppose $x \in \mathbb{R}$. Since $-f(x+p) = -f(x)$, multiplication by -1 gives that $f(x+p) = f(x)$. Therefore, f is periodic. ($\leftarrow$) Suppose f is periodic. Let p be its period. Suppose $x \in \mathbb{R}$. Since $f(x+p) = f(x)$, multiplication by -1 gives that $-f(x+p) = -f(x)$. Therefore, $-f$ is periodic.

4. Let A, B, and C be sets in some universal set $\mathcal{U}$.
   Show: $A \cap (B \setminus C) = (A \setminus C) \cap B$.

   Answer. ($\subseteq$) Suppose $x \in A \cap (B \setminus C)$. So $x \in A$ and $x \in B \setminus C$. Thus, $x \in B$ and $x \notin C$. Since $x \in A$ and $x \notin C$, we have $x \in A \setminus C$. Since we also have $x \in B$, we have $x \in (A \setminus C) \cap B$. ($\supseteq$) Suppose $x \in (A \setminus C) \cap B$. So $x \in A \setminus C$ and $x \in B$. Thus, $x \in A$ and $x \notin C$. Since $x \in B$ and $x \notin C$, we have $x \in B \setminus C$. Since we also have $x \in A$, we have $x \in A \cap (B \setminus C)$.

5. Show: $(0, 2) \cap [1, 3] = [1, 2)$.

   Answer. ($\subseteq$) Suppose $x \in (0, 2) \cap [1, 3]$. That is, $0 < x < 2$ and $1 \leq x \leq 3$. So $1 \leq x < 2$. That is, $x \in [1, 2)$.
   ($\supseteq$) Suppose $x \in [1, 2)$. So $0 < 1 \leq x < 2 \leq 3$. Hence, $0 < x < 2$ and $1 \leq x \leq 3$. That is, $x \in (0, 2) \cap [1, 3]$.

6. Show: $[2, \infty) \times (3, 4] \subseteq (1, 2)^c \times [3, \infty)$.

   Answer. Suppose $(x, y) \in [2, \infty) \times (3, 4]$. So $x \in [2, \infty)$ and $y \in (3, 4]$. Since $x \geq 2$, it follows that $x \notin (1, 2)$. Since $y > 3$, it follows that $y \in [3, \infty)$. Therefore, $(x, y) \in (1, 2)^c \times [3, \infty)$.

7. Let A and B be sets. Show: $\mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$.

   Answer. Suppose $S \in \mathcal{P}(A)$. That is, $S \subseteq A$. Since $A \subseteq A \cup B$, it follows from the transitivity of the subset relation that $S \subseteq A \cup B$. That is, $S \in \mathcal{P}(A \cup B)$.

Section 2.4

1. Show: $\mathbb{Z}$ has no smallest element.

   Answer. Suppose not. Let s be the smallest element of $\mathbb{Z}$. However, $s-1$ is then a smaller element of $\mathbb{Z}$. This is a contradiction.

2. Show: $\mathbb{R}^-$ is infinite.

   Answer. Suppose not. Let n be the cardinality of $\mathbb{R}^-$. However, each of the $n+1$ elements on the list

   $$-1, -2, \ldots, -n, -(n+1)$$

   are negative real numbers. This is a contradiction.

3. Show: $\forall\, n \in \mathbb{Z}, 1 - 2n \neq 0$.

   Answer. Suppose not. So there is some $n \in \mathbb{Z}$ such that $1-2n = 0$. However, this gives that $n = \frac{1}{2}$, and $\frac{1}{2} \notin \mathbb{Z}$. This is a contradiction.

4. Let $a, b \in \mathbb{R}$.
   Show: If $b < a$, then $[a, b] = \emptyset$.

   Answer. Suppose $[a, b]$ is nonempty. Hence, we have some $x \in \mathbb{R}$ such that $a \leq x \leq b$. From the transitivity of $\leq$ it follows that $a \leq b$. Hence, it is not true that $b < a$.

5. Let A and B be sets.
   Show: If $A \subseteq B^c$, then $A \cap B = \emptyset$.

   Answer. Suppose $A \cap B$ is nonempty. So we have some $x \in A \cap B$. That is, $x \in A$ and $x \in B$. Since $x \in A$ and $x \notin B^c$, it follows that $A \nsubseteq B^c$.

6. Let f be a real function.
   Show: If f is unbounded below, then f^2 is unbounded above.

   Answer. Suppose f^2 is bounded above. Hence, we have some $M \in \mathbb{R}$ such that

   $$\forall\, x \in \mathbb{R}, f^2(x) \leq M.$$

   In fact, it must be that $M \geq 0$. It then follows that

   $$\forall\, x \in \mathbb{R}, f(x) \geq -\sqrt{M}.$$

   (This assertion can be proven by contradiction.) Therefore, f is bounded below.

Section 2.5

1. Let A, B, and C be sets. Show: $A \cup C \subseteq A \cup B \cup C$.

   Answer. Suppose $x \in A \cup C$. So $x \in A$ or $x \in C$.
   Case 1: $x \in A$. Since $x \in A$ or $x \in B$ or $x \in C$, it follows that $x \in A \cup B \cup C$.
   Case 2: $x \in C$. Since $x \in A$ or $x \in B$ or $x \in C$, it follows that $x \in A \cup B \cup C$.

2. Let A, B, and C be sets. Show: $(A \cup B) \cap C \subseteq (A \cap C) \cup B$.

   Answer. Suppose $x \in (A \cup B) \cap C$. So $x \in A \cup B$ and $x \in C$. That is, $x \in A$ or $x \in B$.
   Case 1: $x \in A$. Since $x \in A$ and $x \in C$, we have $x \in A \cap C$. Thus, $x \in (A \cap C) \cup B$.
   Case 2: $x \in B$. Thus, $x \in (A \cap C) \cup B$.

3. Let A, B, and C be sets. Show: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

   Answer. ($\subseteq$) Suppose $x \in A \cup (B \cap C)$. So $x \in A$ or $x \in B \cap C$.
   Case 1: $x \in A$. So $x \in A \cup B$ or $x \in A \cup C$. Hence, $x \in (A \cup B) \cap (A \cup C)$.
   Case 2: $x \in B \cap C$. So $x \in B$ and $x \in C$. Since $x \in B$, we have $x \in A \cup B$. Since $x \in C$, we have $x \in A \cup C$. Hence, $x \in (A \cup B) \cap (A \cup C)$.

   ($\supseteq$) Suppose $x \in (A \cup B) \cap (A \cup C)$. So $x \in A \cup B$ and $x \in A \cup C$.
   Case 1: $x \in A$. Hence, $x \in A \cup (B \cap C)$.
   Case 2: $x \notin A$. Since $x \in A \cup B$, it must be that $x \in B$. Since $x \in A \cup C$, it must be that $x \in C$. So, $x \in B \cap C$. Hence, $x \in A \cup (B \cap C)$.

4. Show: $(0, 2) \cup [1, 3] = (0, 3]$.

   Answer. ($\subseteq$) Suppose $x \in (0, 2) \cup [1, 3]$.
   So $x \in (0, 2)$ or $x \in [1, 3]$.
   Case 1: $x \in (0, 2)$. Since $0 < x < 2$, we have $0 < x \leq 3$. Thus, $x \in (0, 3]$.
   Case 2: $x \in [1, 3]$. Since $1 \leq x \leq 3$, we have $0 < x \leq 3$. Thus, $x \in (0, 3]$.
   ($\supseteq$) Suppose $x \in (0, 3]$. So $0 < x \leq 3$.
   (Note: $x < 2$ or $2 \leq x$.)
   Case 1: $x < 2$. Since $0 < x < 2$, we have $x \in (0, 2)$. Thus, $x \in (0, 2) \cup [1, 3]$.
   Case 2: $2 \leq x$. Since $1 \leq 2 \leq x \leq 3$, we have $x \in [1, 3]$. Thus, $x \in (0, 2) \cup [1, 3]$.

5. Assume that $C \subseteq A$ and $C \subseteq B$.
Show: $A \setminus C = B \setminus C$ if and only if $A = B$.

Answer. ($\rightarrow$) Suppose $A \setminus C = B \setminus C$. ($\subseteq$) Suppose $x \in A$.
If $x \in C$, then $x \in B$.
So consider $x \notin C$. Hence, $x \in A \setminus C = B \setminus C$. Thus, $x \in B$. So $A \subseteq B$. Similarly, $B \subseteq A$. Therefore, $A = B$.
($\leftarrow$) Suppose $A = B$. Hence, $A \setminus C = B \setminus C$.

6. Let $x \in \mathbb{R}$. Show: $|x-1| = \begin{cases} x-1 & \text{if } x \geq 1, \\ 1-x & \text{if } x < 1. \end{cases}$

Answer. *Case 1*: $x \geq 1$. Since $x-1 \geq 0$, we have $|x-1| = x-1$.
Case 2: $x < 1$. Since $x-1 < 0$, we have $|x-1| = -(x-1) = 1-x$.

7. Let $x, y \in \mathbb{R}$. Show: If $|x| > y$, then $x > y$ or $x < -y$.

Answer. Suppose $|x| > y$.
Case 1: $x \geq 0$. So $x = |x| > y$.
Case 2: $x < 0$. So $-x = |x| > y$. Multiplication by -1 gives $x < -y$.

8. Let $a, b \in \mathbb{R}$. Show: If $ab > 0$, then $\frac{a}{b} > 0$.

Answer. Suppose $ab > 0$.
Observe that $b \neq 0$.
Case 1: $b > 0$.
So $a > 0$, and hence $\frac{a}{b} > 0$.
Case 2: $b < 0$. So $a < 0$, and hence $\frac{a}{b} > 0$.

3.3 Chapter 3

Section 3.1

1. Show that the sum of any two odd integers is even.

   Answer. Suppose that m and n are odd integers.
   So $m = 2j + 1$ and $n = 2k + 1$ for some $j, k \in \mathbb{Z}$.
   Thus, $m + n = 2j + 1 + 2k + 1 = 2(j + k + 1)$.
   Since $j + k + 1 \in \mathbb{Z}$, the sum $m + n$ is even.

2. Show that the sum of two consecutive odd integers is divisible by 4.

   Answer. Suppose that m and n are consecutive odd integers.
   So $m = 2j + 1$ and $n = 2j + 3$ for some $j \in \mathbb{Z}$.
   Thus, $m + n = 2j + 1 + 2j + 3 = 4(j + 1)$.
   Since $j + 1 \in \mathbb{Z}$, the sum $m + n$ is divisible by 4.

3. Let $n \in \mathbb{Z}$. Show: If $6 \mid n$, then $4 \mid n^2$.

   Answer. Suppose $6 \mid n$. So $n = 6k$ for some $k \in \mathbb{Z}$. Observe that $n^2 = 36k^2 = 4(9k^2)$. Since $9k^2 \in \mathbb{Z}$, we see that $4 \mid n^2$.

4. Prove or disprove: For any integers a, b, c, if $a \nmid b$ and $b \nmid c$ then $a \nmid c$.

   Answer. Counterexample: Let $a = c = 2$ and $b = 3$. Observe that $a \nmid b$ and $b \nmid c$, but $a \mid c$.

5. Find $\gcd(700, 120)$ by factoring.

   Answer. $\gcd(700, 120) = \gcd(2^2 \cdot 5^2 \cdot 7, 2^3 \cdot 3 \cdot 5) = 2^2 \cdot 5 = 20$.

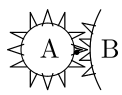
6. Are 3 and 105 relatively prime? Explain

   Answer. No. $\gcd(3, 105) = 3 \neq 1$

7. Let $n \in \mathbb{Z}^+$. Show: $\gcd(n, 2n) = n$.

   Answer. Observe that $n > 0$, $n \mid n$ and $n \mid 2n$. Suppose $c > 0$, $c \mid n$, and $c \mid 2n$. Since c is a divisor of n, there is a lemma that tells us that $c \leq n$. Hence, $n = \gcd(n, 2n)$.

8. Two spinning gear wheels are adjacent, as pictured.

   Gear A has 12 equally-spaced teeth, gear B has n equally-spaced teeth, and the size of B is such that the spacing between its teeth is the same as that of A. What necessary conditions on n force every tooth of gear B to eventually touch the pictured black tooth on gear A?

Answer. $\gcd(12, n) = 1$. That is, $2 \nmid n$ and $3 \nmid n$.

9. Evaluate $\operatorname{lcm}(60, 36)$.

Answer. $\frac{60 \cdot 36}{12} = 180$.

Section 3.2

1. Find the smallest element of the set

$$\{m \ : \ m = 15 + 6n > 0 \text{ for some } n \in \mathbb{Z}\}.$$

Answer. 3. It occurs when $n = -2$.

2. Prove or disprove: If $2^n + 1$ is prime, then n is prime.

Answer. False. Consider $n = 4$.

3. Compute each of the following:

(a) 87 div 12.
(b) 55 mod 7.
(c) -47 mod 10.

Answer.
(a) 7.
(b) 6.
(c) 3.

4. Show: $\forall\, n \in \mathbb{Z}, 4 \nmid (n^2 + 1)$.

Answer. By the Division Algorithm, we can write $n = 4k + r$ for some $k \in \mathbb{Z}$ and $r \in \{0, 1, 2, 3\}$.
So $n^2 + 1 = (4k + r)^2 + 1 = 4k^2 + 8kr + r^2 + 1$.
Case 0: $r = 0$. So, $n^2 + 1 = 4(k^2) + 1$.
Case 1: $r = 1$. So, $n^2 + 1 = 4(k^2 + 2k) + 2$.
Case 2: $r = 2$. So, $n^2 + 1 = 4(k^2 + 4k + 1) + 1$.
Case 3: $r = 3$. So, $n^2 + 1 = 4(k^2 + 6k + 2) + 2$.
In each case, we see that $(n^2 + 1) \bmod 4 = 1$ or 2 (never 0).
Hence, $4 \nmid (n^2 + 1)$.

5. Compute each of the following:

(a) $\lfloor -6.3 \rfloor$.
(b) $\lceil 3.2 \rceil$.
(c) $\lceil -5.8 \rceil$.

Answer.

(a) -7.

(b) 4.

(c) -5.

6. Let $n \in \mathbb{Z}$. Show: $\lfloor n + \frac{1}{2} \rfloor = n$.

   Answer. Observe that $n \in \mathbb{Z}$ and $n \leq n + \frac{1}{2} < n + 1$.

7. Prove or disprove: $\forall\, x, y \in \mathbb{R}, \lfloor xy \rfloor = \lfloor x \rfloor \lfloor y \rfloor$.

   Answer. Counterexample: Let $x = \frac{1}{2}$ and $y = 2$.
   So $\lfloor xy \rfloor = \lfloor 1 \rfloor = 1$ and $\lfloor x \rfloor \lfloor y \rfloor = 0 \cdot 2 = 0$. However, $1 \neq 0$.

8. The identification number $d_1 d_2 \cdots d_{10}$ on an American Express Traveler's Check satisfies $d_1 + d_2 + \cdots + d_{10} \bmod 9 = 0$. Determine the check digit # on the check number 536178450#.

   Answer. 6.

9. The UPC number for Huggies Ultratrim Diapers is read in as

   0 36000 5219# 8,

   where # is a digit that cannot be read. Determine the value of that missing digit.

   Answer. 4.

10. Use the letter to number conversions " " = 0, A = 1, ... , Z = 26 and a shift cipher with $n = 27$ and *encrypting* shift value $b = 5$ to decrypt "HFQHZQZX".

    Answer. "CALCULUS".

11. Use the letter to number conversions " " = 0, A = 1, ... , Z = 26 and a shift cipher with $n = 27$ and *encrypting* shift value $b = 14$ to decrypt "JEB TN FJRW".

    Answer. "WRONG ANSWER".

12. Prove: $\forall\, n \in \mathbb{Z},\ \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$.

    Answer. Case 1: n is even.
    Note $\frac{n}{2}\frac{n}{2} = \frac{n^2}{4}$.
    Case 1: n is odd.
    Note $\frac{n-1}{2}\frac{n+1}{2} = \frac{n^2-1}{4}$.

13. A binary linear code turns a 3-digit binary message $b_1b_2b_3$ into a 6-digit code word $b_1b_2b_3b_4b_5b_6$ according to the following formulas

$$\begin{aligned} b_4 &= (b_1 + b_2 + b_3) \bmod 2 \\ b_5 &= (b_1 + b_2) \bmod 2 \\ b_6 &= (b_2 + b_3) \bmod 2. \end{aligned}$$

  (a) Make a table for the entire code.

  (b) What is the weight of this code?

  (c) Using nearest neighbor decoding, to what message should the code word 111010 be decoded?

Answer. (a)

Message	Code Word
000	000000
001	001101
010	010111
011	011010
100	100110
101	101011
110	110001
111	111100

(b) 3.
(c) 011.

Section 3.3

1. Use any method you wish to find integers x, y such that $\gcd(55, 35) = 55x + 35y$.

   Answer. $\gcd(55, 35) = 5 = 55(2) + 35(-3)$.

2. Prove or disprove that $20x + 16y = 2$ has a solution with $x, y \in \mathbb{Z}$.

   Answer. There is no such solution, since $4(5x + 4y) = 2$ would imply that $4 \mid 2$.

3. Compute $\gcd(68, 20)$ using Euclid's algorithm. Show your work.

   Answer.

$$\begin{aligned} \gcd(68,20) &= \gcd(20,8) && \text{since } 68 = (20)3 + 8 \\ &= \gcd(8,4) && \text{since } 20 = (8)2 + 4 \\ &= \gcd(4,0) && \text{since } 8 = (4)2 + 0 \\ &= 4 && \text{obviously.} \end{aligned}$$

4. Use Euclid's algorithm to find $\gcd(88, 32)$ and to write it in the form $88x + 32y$ for $x, y \in \mathbb{Z}$.

Answer.

$$\begin{array}{rcll} \gcd(88,32) & = & \gcd(32,24) & \text{since } 88=(32)2+24, \quad \text{so } 24=88-(32)2 \\ & = & \gcd(24,8) & \text{since } 32=(24)1+8, \quad \text{so } 8=32-(24)1 \\ & = & \gcd(8,0) & \text{since } 24=(8)3+0 \\ & = & 8 & \text{obviously.} \end{array}$$

Therefore,

$$8 = 32 - (24)1 = 32 - (88 - 2(32))1 = (88)(-1) + (32)(3).$$

That is, $\gcd(88, 32) = 8 = 88x + 32y$ for $x = -1$ and $y = 3$.

5. Show: $\forall\, n \in \mathbb{Z}$, n and $2n + 1$ are relatively prime.

   Answer. $(-2)(n) + (1)(2n + 1) = 1$.

6. Let $m, n, c \in \mathbb{Z}$. Is it always true that, if $c \mid mn$ and $c \nmid m$ then $c \mid n$? Why?

   Answer. No. Consider $m = n = 2$ and $c = 4$.

Section 3.4

1. Show that 1.403 is rational.

   Answer. $1.403 = \frac{1403}{1000}$ and $1403, 1000 \in \mathbb{Z}$ with $1000 \neq 0$.

2. Show that $0.2\overline{34}$ is rational.

   Answer. Let $x = 0.2\overline{34}$. So $10x = 2.\overline{34}$ and $1000x = 234.\overline{34}$.
   Since $990x = 1000x - 10x = 232$, it follows that
   $0.2\overline{34} = x = \frac{232}{990} = \frac{116}{495}$.
   Since $116, 495 \in \mathbb{Z}$ and $495 \neq 0$, we see that $0.2\overline{34}$ is rational.

3. Let $r \in \mathbb{R}$. Use only the definition of $\mathbb{Q}$ to show: If $r \in \mathbb{Q}$, then $\frac{r}{6} \in \mathbb{Q}$.

   Answer. Suppose $r \in \mathbb{Q}$. So $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$.
   Observe that $\frac{r}{6} = \frac{a}{6b}$ and $a, 6b \in \mathbb{Z}$ with $6b \neq 0$. Thus, $\frac{r}{6} \in \mathbb{Q}$.

4. Let $a, b \in \mathbb{Z}$. Show: If $\frac{a}{b}$ is in lowest terms and is positive, then $\frac{b}{a}$ is in lowest terms.

   Answer. Suppose $\frac{a}{b}$ is in lowest terms and is positive. Since $\frac{a}{b}$ is in lowest terms, $\gcd(a, b) = 1$ and $b > 0$. Since $\frac{a}{b}$ is positive, $a > 0$. Since $a > 0$ and $\gcd(b, a) = \gcd(a, b) = 1$, it follows that $\frac{b}{a}$ is in lowest terms.

5. Write $\frac{14}{33}$ in decimal form without using a calculator. Show your work.

Answer.

$$
\begin{array}{rrrrrrl}
 & & & . & 4 & 2 & \\ \cline{2-7}
33 &) & 1 & 4. & 0 & & \\
 & - & 1 & 3 & 2 & & \\ \cline{3-4}
 & & & & 8 & 0 & \text{remainder } 8 \\
 & & & - & 6 & 6 & \\ \cline{4-5}
 & & & & 1 & 4 & \text{remainder } 14
\end{array}
$$

So $\frac{14}{33} = 0.\overline{42}$.

6. Show that $\sqrt{11}$ is irrational by mimicking the proof that $\sqrt{2}$ is irrational.

   Answer. Suppose $\sqrt{11}$ is not irrational. Write $\sqrt{11} = \frac{a}{b}$ in lowest terms. So $b\sqrt{11} = a$. So $b^2 11 = a^2$. So $11 \mid a^2$. Since 11 is prime, $11 \mid a$. Write $a = 11c$. So $b^2 11 = a^2 = 11^2 c^2$. So $b^2 = 11c^2$. So $11 \mid b^2$. Since 11 is prime, $11 \mid b$. So $\gcd(a, b) \geq 11$. This contradicts the assumption that $\frac{a}{b}$ is in lowest terms.

7. Use the result from problem 1 to show that $\frac{3}{\sqrt{11}-1}$ is irrational.

   Answer. Suppose to the contrary that $r = \frac{3}{\sqrt{11}-1}$ is rational. So $\sqrt{11} = \frac{r+3}{r}$. Since r is clearly nonzero, the arithmetic properties of $\mathbb{Q}$ guarantee that $\frac{r+3}{r}$ is rational. This contradicts the result from problem 1 that $\sqrt{11}$ is irrational. Hence, it must be that $\frac{3}{\sqrt{11}-1}$ is irrational.

8. Show that $\log_2(11) \notin \mathbb{Q}$.

   Answer. Suppose to the contrary that $\log_2(11) \in \mathbb{Q}$. Write $\log_2(11) = \frac{a}{b}$, with $a, b > 0$. So $2^{\frac{a}{b}} = 11$. So $2^a = 11^b$. By the Fundamental Theorem of Arithmetic, $a = b = 0$. This is a contradiction.

9. Is the product of irrational numbers always irrational? Justify your answer.

   Answer. No. E.g. $x = y = \sqrt{2}, xy = 2$.

10. Show: $\forall\, x \in \mathbb{R}$, if $x^2 \notin \mathbb{Q}$, then $x \notin \mathbb{Q}$.

    Answer. Suppose $x \in \mathbb{Q}$. Hence, the product $x^2 \in \mathbb{Q}$.

11. Use the Rational Roots Theorem to show that $\sqrt{1+\sqrt{2}} \notin \mathbb{Q}$.

Answer. Scratch work.
Let $r = \sqrt{1+\sqrt{2}}$.
So $r^2 = 1+\sqrt{2}$.
Hence, $r^2 - 1 = \sqrt{2}$.
So $r^4 - 2r^2 + 1 = 2$.
Therefore, $r^4 - 2r^2 - 1 = 0$.
Proof.
Let $f(x) = x^4 - 2x^2 - 1$. By the Rational Roots Theorem, the only possible rational roots of f are ± 1. Of course, $\sqrt{1+\sqrt{2}}$ is neither 1 nor -1. Since $f(\sqrt{1+\sqrt{2}}) = 0$, it follows that $\sqrt{1+\sqrt{2}}$ must be irrational.

12. Show that $\frac{1}{\sqrt{2}+\sqrt{3}}$ is algebraic.

Answer. Let $x = \frac{1}{\sqrt{2}+\sqrt{3}} = \sqrt{3} - \sqrt{2}$. So $x^2 = 5 - 2\sqrt{6}$. Since $x^2 - 5 = -2\sqrt{6}$, we see that $x^2 - 10x + 25 = (x^2-5)^2 = 24$. Thus, $x^2 - 10x + 1 = 0$. Since $\frac{1}{\sqrt{2}+\sqrt{3}}$ is a root of the polynomial $x^2 - 10x + 1$ (which has integer coefficients), $\frac{1}{\sqrt{2}+\sqrt{3}}$ is algebraic.

Section 3.5

1. Determine if the following statements are True or False.

   (a) $28 \equiv 10 \pmod 3$.
   (b) $4 \equiv 0 \pmod 8$.

   Answer.
   (a) True.
   (b) False.

2. In a single year, is it possible for July 4th and Christmas (December 25th) to occur on the same day of the week? Justify your answer.

   Answer. No, since $(27+31+30+31+30+25) \bmod 7 = 6 \neq 0$.

3. Let $a, b, n \in \mathbb{Z}$ with $n > 1$.
   Show: If $a \equiv -b \pmod n$, then $a^2 \equiv b^2 \pmod n$.

   Answer. Suppose $a \equiv -b \pmod n$. So $n \mid (a+b)$. That is, $(a+b) = nk$ for some $k \in \mathbb{Z}$. Observe that

   $$a^2 - b^2 = (a+b)(a-b) = n \cdot k(a-b)$$

   and $k(a-b) \in \mathbb{Z}$. Thus, $n \mid a^2 - b^2$. That is, $a^2 \equiv b^2 \pmod n$.

4. Compute $(3^{68135} + 35) \bmod 9$.

Answer. Observe that

$$3^{68135} = 9 \cdot 3^{68133} \equiv 0 \cdot 3^{68133} \equiv 0 \pmod 9.$$

Hence, $(3^{68135} + 35) \bmod 9 = 35 \bmod 9 = 8$.

5. Use the fact that $3^4 \equiv 1 \pmod{10}$ to compute $3^{53186} \bmod 10$.

   Answer. $3^{53186} \equiv (3^4)^{13296} \cdot 3^2 \equiv 1^{13296} \cdot 9 \equiv 9 \pmod{10}$. So, $3^{53186} \bmod 10 = 9$.

6. Show: $\forall\, n \in \mathbb{Z}, (3n^4+1)^2 \equiv 1 \pmod 5$.

   Answer. Let $n \in \mathbb{Z}$. First observe that $n^4 \equiv \begin{cases} 0 & \text{if } n \equiv 0 \pmod 5, \\ 1 & \text{if } n \not\equiv 0 \pmod 5. \end{cases}$
   So, $3n^4 + 1 \equiv \begin{cases} 1 & \text{if } n \equiv 0 \pmod 5, \\ -1 & \text{if } n \not\equiv 0 \pmod 5. \end{cases}$
   Since $1^2 = (-1)^2 = 1$, we see that $(3n^4+1)^2 \equiv 1 \pmod 5$.

7. Use Fermat's Little Theorem to help you compute $7^{123432} \bmod 11$.

   Answer. Since 11 is prime and $11 \nmid 7$, Fermat's Little Theorem tells us that $7^{10} \equiv 1 \pmod{11}$. Hence,

   $$7^{123432} \equiv (7^{10})^{12343} \cdot 7^2 \equiv 1^{12343} \cdot 49 \equiv 49 \equiv 4 \pmod{11}.$$

   Therefore, $7^{123432} \bmod 11 = 4$.

8. A certain product ID code is 4 characters long and is constructed using only the letters in Table 3.1. A linear cipher with $n = 7$, $a = 3$, and $b = 1$

A	B	C	D	E	F	G
0	1	2	3	4	5	6

Table 3.1: Converting Letters to Numbers

   (i.e. $y = (3x + 1) \bmod 7$) is used to encode the ID's.

   (a) Encrypt 'FACE'.
   (b) Decrypt 'GDFG'.

   Answer.
   (a) CBAG.
   (b) EDGE.

9. Use binary expansion and repeated squaring to compute $20^7 \bmod 403$.

Answer. 266.
Note that $7 = 4 + 2 + 1$, and $20^2 \equiv -3$, $20^4 \equiv 9 \pmod{403}$.
Also, $9 \cdot (-3) \cdot 20 \equiv -540 \equiv 266 \pmod{403}$.

10. A company is using the RSA encryption method with $p = 7$ and $q = 17$, so $n = 119$. The number $a = 35$ is used to encode messages via $y = x^{35} \bmod 119$. Note that $c = 11$ is a multiplicative inverse of a modulo 48.

    (a) Encrypt the message $x = 2$.

    (b) Decrypt the message $y = 5$.

    Answer.
    (a) 25. (Note that $2^7 \equiv 9 \pmod{119}$.)
    (b) 45. (Note that $5^3 \equiv 6 \pmod{119}$.)

11. Find and simplify $[13]_{10} + [7]_{10}$.

    Answer. $[20]_{10} = [0]_{10}$.