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## Chapter 2 Solutions

### Section 2.1

1. Some possibilities:

(a)  $a_n = [a + b + (-1)^n(b - a)]/2.$

(b)  $a_n = [a + b + (-1)^{\lfloor (n+1)/2 \rfloor}(b - a)]/2,$   
 $a_n = [a + b + (a - b)[\sin(n\pi/2) - \cos(n\pi/2)]]/2.$

(c)  $a_n = [a + b + (-1)^{\lfloor (n-1)/3 \rfloor}(a - b)]/2.$

(d)  $a_n = \frac{1}{2}(b + c - 2a)x_n^2 + \frac{1}{2}(b - c)x_n + a, x_n := \sin[(n - 1)\pi/2].$

(e)  $a_n = 3 + (-1)^{\lfloor (n+1)/2 \rfloor} + [(-1)^n - 1]/2.$

2.  $x_1 = a, x_n = a + b - x_{n-1}, n > 1.$

3. (a) Since  $|(4n - 1)/(2n + 7) - 2| = 15/(2n + 7) < 8/n$ , choose any integer  $N \geq 8/\varepsilon$ .

(b) If  $n \geq 6$ ,  $|(2n^2 - n)/(n^2 + 3) - 2| = |n + 6|/(n^2 + 3) \leq 2n/n^2 = 2/n$ . Therefore, choose  $N \geq \min\{6, 2/\varepsilon\}$ .

(c)  $|(5\sqrt{n} + 7)/(3\sqrt{n} + 2) - 5/3| = 11/(9\sqrt{n} + 6) < 11/\sqrt{n}$ , so choose any integer  $N \geq (11/\varepsilon)^2$ .

(d) For  $n \geq 2$ ,  $(n - 1)/(\sqrt{n} + 1) \geq (n/2)/2\sqrt{n} = \sqrt{n}/4$ , so choose any integer  $N \geq 16M^2$ .

(e)  $|(2 + 1/n)^3 - 8| = [(2 + 1/n)^2 + 2(2 + 1/n) + 4]/n \leq 19/n$ , so choose any integer  $N > 19/\varepsilon$ .

(f)  $\sqrt{\frac{n+2}{n+1}} - 1 = \frac{1}{\sqrt{n+1}(\sqrt{n+2} + \sqrt{n+1})} \leq \frac{1}{n}$ , so choose any integer  $N > 1/\varepsilon$ .

4. The disjoint intervals  $(-3/2, -1/2)$  and  $(1/2, 3/2)$  each contain infinitely many terms of the sequence. Therefore, no limit can exist.

5. Let  $r = pq^{-1}$ ,  $p, q \in \mathbb{Z}$ ,  $q > 0$ . For all  $n \geq q$ ,  $n!r \in \mathbb{Z}$  hence  $\sin(n!r\pi) = 0$ .

6. The general term in the sequence may be written  $n^{p-1}(1 + n^{-2})^p$ , which tends to 1 if  $p = 1$ , 0 if  $p < 1$ , and  $+\infty$  if  $p > 1$ .

7. Let  $A = \{x_1, \dots, x_p\}$  and  $A_j = \{n : a_n = x_j\}$ . One of these sets, say  $A_1$ , must have infinitely many members. Since  $|x_1 - a| \leq |x_1 - a_n| + |a_n - a|$  and  $a_n \rightarrow a$ , letting  $n \rightarrow +\infty$  through  $A_1$  shows that  $x_1 = a$ . We may therefore choose  $\varepsilon > 0$  so that  $I := (a - \varepsilon, a + \varepsilon)$  contains no  $x_j$  for  $j \geq 2$ . Let  $N \in \mathbb{N}$  such that  $a_n \in I$  for all  $n \geq N$ . For such  $n$ ,  $a_n = a$ .

8. (a)  $b_n = (3a_n + 2b_n - 3a_n)/2 \rightarrow (c - 3a)/2$ .

(b) Let  $c_n = 3a_nb_n + 5a_n^2 - 2b_n$ . Then

$$b_n = (c_n - 5a_n^2)/(3a_n - 2) \rightarrow (1 - 20)/(6 - 2) = -19/4.$$

9. (a) 2. (b)  $\sqrt{a/b}$ . (c)  $k/2$ . (d)  $b/2\sqrt{a}$ . (e) 1. (f)  $1/2a$ . (g)  $-ka^{k-1}$ .  
(h)  $a/k$ . (i) 0. (j) 0. (k)  $1/2$ . (l) 1.

10. If  $|a_n| \leq M$  for all  $n$ , then  $|a_nb_n| \leq M|b_n| \rightarrow 0$ .

11. Use  $-r \leq a_n - b_n \leq r$  and 2.1.4.

12.  $\sqrt{n}a_n = (na_n)(1/\sqrt{n}) \rightarrow a \cdot 0 = 0$ .

13. If  $a = 0$ , given  $\varepsilon > 0$  choose  $N$  such that  $a_n < \varepsilon^k$  for all  $n \geq N$ . Suppose  $a > 0$ . Then there exists  $N$  such that  $a_n > 0$  for all  $n \geq N$ . By Exercise 1.4.15,

$$|a_n^{1/k} - a^{1/k}| = |a_n - a| \left( \sum_{j=1}^k a_n^{1-j/k} a^{(j-1)/k} \right)^{-1} \rightarrow 0,$$

since the expression inside the parentheses tends to

$$\sum_{j=1}^k a^{1-j/k} a^{(j-1)/k} = ka^{1-1/k} > 0.$$

Therefore,  $a_n^{1/k} \rightarrow a^{1/k}$ .

14. (a) Suppose first that  $r > 1$ . Set  $h_n = r^{1/n} - 1$ . Then  $h_n > 0$ , and by the binomial theorem,  $r = (1 + h_n)^n > nh_n$ . Therefore, by the squeeze principle,  $h_n \rightarrow 0$ . If  $r < 1$  consider  $1/r$ .

(b) Set  $h_n = n^{1/n} - 1$ . Then  $n = (1 + h_n)^n > n(n-1)h_n^2/2$ , hence  $h_n \rightarrow 0$ .

(c) Set  $h_n = (r + n^k)^{1/n} - 1$ . By the binomial theorem, for  $n \geq k$

$$r + n^k = (1 + h_n)^n > \frac{n(n-1) \cdots (n-k)h_n^{k+1}}{(k+1)!} > \frac{(n-k)^{k+1}h_n^{k+1}}{(k+1)!},$$

hence  $h_n \rightarrow 0$ .

(d) Use the inequality  $2x/\pi \leq \sin x \leq x$ ,  $0 \leq x \leq \pi/2$ , and the squeeze principle.

15. Follows from the identities  $x = x^+ - x^-$ ,  $x^+ = (|x| + x)/2$ , and  $x^- = (|x| - x)/2$ .

16. Let  $s = 1/|r|$  and  $h = s - 1$ . By the binomial theorem,

$$s^n = (h + 1)^n = \sum_{k=0}^n \binom{n}{k} h^k.$$

Since  $s > 1$ , each term in the sum is positive hence, for  $n > m$ ,

$$s^n > \binom{n}{m+1} h^{m+1} = \frac{n(n-1)\cdots(n-m)}{(m+1)!} h^{m+1} > \frac{(n-m)^{m+1}}{(m+1)!} h^{m+1}.$$

Therefore,

$$0 < |n^m r^n| = \frac{n^m}{s^n} < \frac{n^m(m+1)!}{(n-m)^{m+1}h^{m+1}} = \frac{(m+1)!}{n(1-m/n)^{m+1}h^{m+1}}.$$

Since the term on the right tends to 0 as  $n \rightarrow +\infty$ , the squeeze principle implies that  $n^m r^n \rightarrow 0$ .

17.  $a^n < r a_{n-1} < r^2 a_{n-2} < \cdots < r^{n-1} a_1 \rightarrow 0$ . For the example, take  $a_n = 2^{1/n}$ .
18. Suppose first that  $a \in \mathbb{R}$ . Given  $\varepsilon > 0$ , choose  $N$  such that  $|a_n - a| < \varepsilon/2$  for all  $n > N$ . For such  $n$ ,

$$\begin{aligned} \left| \frac{a_1 + \cdots + a_n}{n} - a \right| &\leq \left| \frac{(a_1 - a) + \cdots + (a_N - a)}{n} \right| \\ &\quad + \left| \frac{(a_{N+1} - a) + \cdots + (a_n - a)}{n} \right| \\ &\leq \left| \frac{(a_1 - a) + \cdots + (a_N - a)}{n} \right| + \frac{n - N}{n} \frac{\varepsilon}{2}. \end{aligned}$$

The second term on the right in the last inequality is less than  $\varepsilon/2$ . Also, there exists  $N' > N$  such that the first term is less than  $\varepsilon/2$  for all  $n \geq N'$ . For such  $n$ ,  $|(a_1 + \cdots + a_n)/n - a| < \varepsilon$ .

Now suppose  $a_n \rightarrow +\infty$ . Let  $M > 0$  and choose  $N$  such that  $a_n > 4M$  for all  $n > N$ . For such  $n$ ,

$$\begin{aligned} \frac{a_1 + \cdots + a_n}{n} &= \frac{a_1 + \cdots + a_N}{n} + \frac{a_{N+1} + \cdots + a_n}{n} \\ &\geq \frac{a_1 + \cdots + a_N}{n} + \frac{4(n - N)M}{n}. \end{aligned}$$

Choose  $N' > N$  such that

$$\frac{n - N}{n} > \frac{1}{2} \quad \text{and} \quad \frac{a_1 + \cdots + a_N}{n} > -M$$

for all  $n \geq N'$ . For such  $n$ ,  $(a_1 + \cdots + a_n)/n \geq 2M - M = M$ .

The converse is false: consider  $a_n = (-1)^n$ .

19. Choose  $N$  such that  $a_n - a < \varepsilon$  for all  $n \geq N$ . For such  $n$ ,

$$0 \leq \min\{a_1, \dots, a_n\} - a \leq a_n - a < \varepsilon.$$

Therefore,  $\min\{a_1, \dots, a_n\} \rightarrow a$ . The converse is false: consider  $a_n = 1 + (-1)^n$ .

20. Given  $\varepsilon > 0$ , choose  $N$  such that  $|a_n|/n < \varepsilon$  for all  $n \geq N$ . Then

$$b_n := n^{-1} \max\{a_1, \dots, a_n\} = \max\{\alpha_n, \beta_n\},$$

where

$$\alpha_n := n^{-1} \max\{a_1, \dots, a_N\}, \quad \beta_n = n^{-1} \max\{a_{N+1}, \dots, a_n\}$$

Choose  $N' > N$  such that  $|\alpha_n| < \varepsilon$  for all  $n \geq N'$ . For such  $n$  we also have  $-\varepsilon < \beta_n < \varepsilon$ , hence  $-\varepsilon < b_n < \varepsilon$ .

If  $\{a_n\}$  is bounded below by  $c$  then

$$c/n \leq a_n/n \leq \max\{a_1, \dots, a_n\}/n.$$

Hence if  $(1/n) \max\{a_1, \dots, a_n\} \rightarrow 0$ , then  $a_n/n \rightarrow 0$ . The example  $a_n = 1 - n$  shows that the converse is not generally true.

21.  $(x_1^n + \dots + x_k^n)^{1/n} = x_k [(x_1/x_k)^n + \dots + (x_{k-1}/x_k)^n + 1]^{1/n}$  and

$$1 \leq [(x_1/x_k)^n + \dots + (x_{k-1}/x_k)^n + 1]^{1/n} \leq k^{1/n} \rightarrow 1.$$

22. Suppose that  $c \leq f(x) - x \leq d$  for all  $x$ , so  $c + jx \leq f(jx) \leq dx$ . Summing and using Exercise 1.5.4,

$$nc + xn(n+1)/2 \leq \sum_{j=1}^n f(jx) \leq nd + xn(n+1)/2$$

hence

$$c/n + x(1 + 1/n)/2 \leq (1/n^2) \sum_{j=1}^n f(jx) \leq d/n + x(1 + 1/n)/2.$$

Letting  $n \rightarrow +\infty$ , we obtain (a). Part (b) is proved similarly.

23. Let  $c = a_1/a_0$  and  $r = -1/2$ . By induction,  $\frac{a_{n+1}}{a_n} = c^{r^n}$  hence

$$a_{n+1} = \left(\frac{a_{n+1}}{a_n}\right) \dots \left(\frac{a_1}{a_0}\right) a_0 = a_0 c^{1+r+\dots+r^n} \rightarrow a_0 c^{1/(1-r)} = a_0^{1/3} a_1^{2/3}$$

24. Given  $\varepsilon > 0$  choose  $N$  such that  $|a_{n+k} - a_n - c| < \varepsilon$  for all  $n \geq N$ . Let  $n \geq N+k$  and choose  $q_n, r_n \in \mathbb{Z}$  such that  $n - N = q_n k + r_n$ ,  $0 \leq r_n < k$  (division algorithm). Then  $q_n = k^{-1}(n - N - r_n) \rightarrow +\infty$  and

$$a_n - q_n c = \sum_{j=1}^{q_n} (a_{n-(j-1)k} - a_{n-jk} - c) + a_{n-q_n k}$$

Since  $n - jk \geq n - q_n k \geq N$ , the terms of the sum have absolute value less than  $\varepsilon$ . Thus for all large  $n$ ,

$$\left| \frac{a_n}{q_n} - c \right| = \frac{1}{q_n} |a_n - q_n c| < \varepsilon + \frac{a_{n-q_n k}}{q_n} = \varepsilon + \frac{a_{N+r_n}}{q_n}$$

so  $a_n/q_n \rightarrow c$ . Since  $a_n/n = (a_n/q_n)(q_n/n)$  and  $q_n/n \rightarrow 1$ ,  $a_n/n \rightarrow c$ .

## Section 2.2

1. Since

$$\frac{a^{1/n}}{a^{1/(n+1)}} = a^{1/n(n+1)} < 1 < b^{1/n(n+1)} = \frac{b^{1/n}}{b^{1/(n+1)}},$$

$a^{1/n}$  is increasing and  $b^{1/n}$  is decreasing. Each tends to 1 by Exercise 2.1.14.

2. Since  $a < 1$ ,  $a^{n+1}/(n+1)^k < a^n/n^k$ . For large  $n$ ,  $b > (n+1)^k/n^k$ , hence  $b^{n+1}/(n+1)^k > b^n/n^k$ .
3. By results of Section 2.1,

$$a_n = a(1/n + nb)^{-1} \rightarrow 0 \quad \text{and} \quad na_n = a(1/n^2 + b)^{-1} \rightarrow ab^{-1}.$$

The condition  $a_{n+1} < a_n$  is equivalent to  $(n^2 + n)b > 1$ , which holds eventually. Similarly,  $(n+1)a_{n+1} > na_n$  is equivalent to the inequality  $(n+1)^2 > n^2$ .

4.  $(x_1^n + \cdots + x_n^n)^{1/n} = x_n [(x_1/x_n)^n + \cdots + (x_{n-1}/x_n)^n + 1]^{1/n}$  and  $1 \leq [(x_1/x_n)^n + \cdots + (x_{n-1}/x_n)^n + 1]^{1/n} \leq n^{1/n} \rightarrow 1$ .
5. Let  $r_n$  be any strictly increasing sequence converging to  $\sup A$ . By the approximation property, there exists  $a_1 \in A$  with  $r_1 < a_1 \leq \sup A$ ,  $a_2 \in A$  with  $r_2 < a_2 \leq \sup A$  and  $a_2 \geq a_1$ , etc. In this way we obtain a sequence  $a_1 \leq a_2 \leq \cdots$  converging to  $\sup A$ .
6. Suppose  $a_n$  is increasing. Then

$$\begin{aligned} b_{n+1} - b_n &= \frac{a_1 + \cdots + a_{n+1}}{n+1} - \frac{a_1 + \cdots + a_n}{n} \\ &= \frac{n(a_1 + \cdots + a_{n+1}) - (n+1)(a_1 + \cdots + a_n)}{n(n+1)} \\ &= \frac{na_{n+1} - (a_1 + \cdots + a_n)}{n(n+1)} \geq 0. \end{aligned}$$

7. Let  $f(x) = 1 + \frac{1}{2 + (1+x)^{-1}} = \frac{3x+4}{2x+3}$ . Then  $f : [1, 2] \rightarrow [1, 2]$ ,  $f$  is increasing and  $f(a_m) = a_{m+2}$ . Since  $a_1, a_2 \in [1, 2]$ ,  $a_n \in [1, 2]$  for all  $n$ . Since  $a_1 = 1$ ,  $a_2 = 3/2$ ,  $a_3 = 7/5$  and  $a_4 = 17/12$ , the inequalities

$$a_{2n+2} < a_{2n} \quad \text{and} \quad a_{2n+1} > a_{2n-1}$$

hold for  $n = 1$ . Assume they hold for  $n = k$ . Then

$$\begin{aligned} a_{2k+4} &= f(a_{2k+2}) < f(a_{2k}) = a_{2k+2} \quad \text{and} \\ a_{2k+3} &= f(a_{2k+1}) > f(a_{2k-1}) = a_{2k+1} \end{aligned}$$

hence the inequalities hold for  $n = k + 1$ .

Since the sequences  $\{a_{2n}\}$  and  $\{a_{2n+1}\}$  are bounded and monotone, the monotone convergence theorem implies that  $a_{2n} \rightarrow a$  and  $a_{2n+1} \rightarrow b$  for some  $a, b \in \mathbb{R}$ . Letting  $n \rightarrow +\infty$  in  $f(a_{2n}) = a_{2n+2}$  gives  $f(a) = a$ . Therefore,  $a = \sqrt{2}$ . Similarly,  $b = \sqrt{2}$ . Therefore,  $a_n \rightarrow \sqrt{2}$ .

8.  $a_1 = \sqrt{r + \sqrt{r}} > \sqrt{r} = a_0$ , and if  $a_n > a_{n-1}$  then

$$a_{n+1} = a_n = \sqrt{r + a_n} > \sqrt{r + a_{n-1}} > a_n$$

Therefore, by induction,  $\{a_n\}$  is strictly increasing. Also,  $a_0 < \sqrt{r} + 1$ , and if  $a_n < \sqrt{r} + 1$  then

$$a_{n+1} = \sqrt{r + a_n} < \sqrt{r + \sqrt{r} + 1} < \sqrt{r} + 1.$$

Therefore,  $\{a_n\}$  is bounded above by  $\sqrt{r} + 1$ . By the monotone convergence theorem,  $a_n \rightarrow a$  for some  $a \in \mathbb{R}$ . Letting  $n \rightarrow +\infty$  in  $a_n = \sqrt{r + a_{n-1}}$  produces  $a = \sqrt{r + a}$ , which has positive solution  $a = (1 + \sqrt{1 + 4r})/2$ .

9. For  $x > 0$ ,  $x^2 + r \geq 2x\sqrt{r}$  hence  $(x + r/x)/2 \geq \sqrt{r}$ . Therefore,  $a_n \geq \sqrt{r}$ . For  $x \geq \sqrt{r}$ ,  $x^2 + r \leq 2x^2$  hence  $(x + r/x)/2 \leq x$ . Therefore,  $a_n \geq a_{n+1}$ . By the monotone convergence theorem,  $a_n \rightarrow a$  for some  $a \geq \sqrt{r}$ . Letting  $n \rightarrow +\infty$  in  $a_n = (a_{n-1} + r/a_{n-1})/2$ , yields  $a = (a + r/a)/2$ , which has positive solution  $a = \sqrt{r}$ .
10. Let  $a_n := (1 - 1/n^2)^n = (1 - 1/n)(1 + 1/n)$ . By Bernoulli's inequality,  $1 - 1/n \leq a_n \leq 1$ , hence  $a_n \rightarrow 1$  and so  $(1 - 1/n)^n = a_n/(1 + 1/n)^n \rightarrow 1/e$ .

Alternatively,

$$\left[1 - \frac{1}{n}\right]^{-n} = \left[\frac{n}{n-1}\right]^n = \left[1 + \frac{1}{n-1}\right]^n = \left[1 + \frac{1}{n-1}\right] \left[1 + \frac{1}{n-1}\right]^{n-1},$$

which tends to  $e$ .

11. Let  $x, y > 0$ . Since  $(x - y)^2 \geq 0$ ,  $\sqrt{xy} \leq (x + y)/2$ , with strict equality holding iff  $x \neq y$ . Also,  $0 < x < y$  implies  $\sqrt{xy} > x$  and  $(x + y)/2 < y$ .

Now let  $P_n$  be the statement  $0 < x_n < x_{n+1} < y_{n+1} < y_n$ . From the above discussion,  $P_0$  is true, and  $P_n$  implies  $P_{n+1}$ . Therefore, the sequences  $\{x_n\}$  and  $\{y_n\}$  are monotone and bounded. Let  $x_n \uparrow x$  and  $y_n \downarrow y$ , so  $0 < x \leq y$ . Letting  $n \rightarrow +\infty$  in

$$y_{n+1} - x_{n+1} \left( = (x_n + y_n)/2 - \sqrt{x_n y_n} \right) = (\sqrt{y_n} - \sqrt{x_n})^2 / 2$$

yields

$$(\sqrt{y} + \sqrt{x})(\sqrt{y} - \sqrt{x}) = y - x = (\sqrt{y} - \sqrt{x})^2 / 2.$$

It follows that  $y = x$ .

### Section 2.3

1. (a)  $0, \pm 3/8$ . (b)  $0, \pm 1, \pm 2$ . (c)  $\pm 4, \pm 6, \pm 12, \pm 14$ . (d)  $0, 3, \pm 1$ .  
 2. For example,  $1, 2, 3, 1, 1, 2, 3, 2, 1, 2, 3, 3, \dots, 1, 2, 3, n, \dots$   
 3. (a)  $e^{1/k}$ . (b)  $e$ . (c)  $0$  ( $k \geq 2$ ). (d)  $e^{k/2}$ . (e)  $e^{7/3}$ .

For example, for (d)

$$a_n^{2/k} = \left(1 + \frac{1}{2n+k}\right)^{2n+k} \left(1 + \frac{1}{2n+k}\right)^{-k} \rightarrow e.$$

4. By Bolzano-Weierstrass, there exists a convergent subsequence  $\{a_{n_k}\}$  of  $\{a_n\}$ . Similarly, there exists a convergent subsequence  $\{b_{n_{k_j}}\}$  of  $\{b_{n_k}\}$ .  
 5. If  $\{a_n\}$  lies in the set  $\{x_1, \dots, x_n\}$ , then one of the sets  $\{n : a_n = x_j\}$  must have infinitely many members and a subsequence may be constructed from these.  
 6. Let  $\{r_n\}$  be any strictly increasing sequence with limit  $r$ . Choose  $n_1$  such that  $a_{n_1} > r_1$ ,  $n_2 > n_1$  such that  $a_{n_2} > \max\{r_2, a_{n_1}\}$ , and in general choose  $n_k > n_{k-1}$  such that  $a_{n_k} > \max\{r_k, a_{n_{k-1}}\}$ .  
 7. We may assume that  $a_n \rightarrow a \in \overline{\mathbb{R}}$  (otherwise take a subsequence). Either  $a_n < a$  for infinitely many  $n$  or  $a_n > a$  for infinitely many  $n$ . Assume the former. Choose  $n_1$  such that  $a_{n_1} < a$ . Since there are infinitely many  $n$  for which  $a_{n_1} < a_n < a$ , we may choose  $n_2 > n_1$  such that  $a_{n_1} < a_{n_2} < a$ , etc.  
 8. Given  $\varepsilon > 0$ , choose  $N$  so that  $\sum_{n=N}^{\infty} |a_{n+k} - a_n| < \varepsilon$ . For  $m > n \geq N$ ,

$$|a_{mk} - a_{nk}| \leq |a_{mk} - a_{(m-1)k}| + \dots + |a_{(n+1)k} - a_{nk}| < \varepsilon.$$

Therefore,  $\{a_{nk}\}_{n=1}^{\infty}$  is Cauchy.

9.  $|a_{n+1} - a_n| = |a_n - a_{n-1}|/2 = \cdots = |a_1 - a_0|/2^n$  hence for  $m > n$

$$|a_m - a_n| \leq |a_m - a_{m-1}| + \cdots + |a_{n+1} - a_n| < |a_1 - a_0| \sum_{k=n}^{\infty} 2^{-k}.$$

Since the series converges,  $\{a_n\}$  is Cauchy.

10. Clearly  $a_n \rightarrow 0$  implies  $b_n \rightarrow 0$ . For the converse, note that

$$b_n = \frac{1}{a_n^{-q} + a_n^{p-q}} \leq a_n^q.$$

If  $0 < q < p$ , then the sufficiency is false: Take  $a_n = n$ ,  $q = 1/2$  and  $p = 1$ . Then  $b_n = \sqrt{n}/(n+1) \rightarrow 0$  but  $a_n \rightarrow +\infty$ .

11. For  $x \in I$ , choose  $n_1$  such that  $a_{n_1} \in (x-1, x+1)$ , then choose  $n_2 > n_1$  such that  $a_{n_2} \in (x-1/2, x+1/2)$ , and in general choose  $n_k > n_{k-1}$  such that  $a_{n_k} \in (x-1/k, x+1/k)$ . Then  $a_{n_k} \rightarrow x$ .

For the example, take  $\{a_n\}$  to be an enumeration of the rationals.

12. First, choose a subsequence  $\{b_{m_k}\}$  such that  $|b_{m_k} - b| < 1/k$ . Then choose  $n_1$  such that  $|a_{n_1} - b_{m_1}| < 1$ ,  $n_2 > n_1$  such that  $|a_{n_2} - b_{m_2}| < 1/2$ , and in general choose  $n_k > n_{k-1}$  such that  $|a_{n_k} - b_{m_k}| < 1/k$ . Then  $|a_{n_k} - b| < 2/k$  for all  $k$  so  $a_{n_k} \rightarrow b$ .

## Section 2.4

1. (a)  $\liminf_n = -5/3$ ,  $\limsup_n = 5/3$ .  
 (b)  $\liminf_n = 0$ ,  $\limsup_n = +\infty$ .  
 (c)  $\liminf_n = -14$ ,  $\limsup_n = 14$ .  
 (d)  $\liminf_n = 1$ ,  $\limsup_n = 4$ .  
 (e)  $\liminf_n = x + y - z$ ,  $\limsup_n = -x + y + z$ .  
 (f)  $\liminf_n = ar^2/(1-r)$ ,  $\limsup_n = ar/(1-r)$ .  
 (g)  $\liminf_n = 0$ ,  $\limsup_n = +\infty$ .  
 (h)  $\liminf_n = -\infty$ ,  $\limsup_n = +\infty$ .
2. (b) and (c):  $a_n = (-1)^n$ ,  $b_n = (-1)^{n+1}$ ;  
 (f) and (g):  $a_n = 2 + (-1)^n$ ,  $b_n = 2 - (-1)^n$ .
3. Follows from Exercise 1.4.6.
4. Follows from Exercise 1.4.6.
5. Follows from  $\{a_{n_k} : k \geq n\} \subseteq \{a_k : k \geq n\}$ .



$$6. \quad 0 < b - \varepsilon < b_n < b + \varepsilon \Rightarrow a_n + b - \varepsilon < a_n + b_n < a_n + b + \varepsilon \Rightarrow$$

$$b - \varepsilon + \limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} (a_n + b_n) \leq b + \varepsilon + \limsup_{n \rightarrow \infty} a_n.$$

Now let  $\varepsilon \rightarrow 0$ .

$$7. \quad 0 < b - \varepsilon < b_n < b + \varepsilon \Rightarrow a_n(b - \varepsilon) < a_n b_n < a_n(b + \varepsilon) \Rightarrow$$

$$(b - \varepsilon) \limsup_n a_n \leq \limsup_n a_n b_n \leq (b + \varepsilon) \limsup_n a_n.$$

Now let  $\varepsilon \rightarrow 0$ .

8. If  $a_{n_k} \rightarrow \bar{a} := \limsup_n a_n$ , then  $|a_{n_k}| \rightarrow |\bar{a}|$ . Therefore,  $|\bar{a}|$  is a limit point of  $|a_n|$  and the result follows from 2.4.2. The sequence  $a_n = (-1)^n - 1$  shows that the inequalities may be strict.

9. For each  $N$  choose  $K$  so that  $\{1, \dots, N-1\} \subseteq \{n_k : 1 \leq k \leq K-1\}$ . Then  $k \geq K \Rightarrow n_k \geq N$  hence

$$\limsup_k a_{n_k} \leq \sup_{k \geq K} a_{n_k} \leq \sup_{n \geq N} a_n.$$

Letting  $N \rightarrow +\infty$  yields  $\limsup_k a_{n_k} \leq \limsup_n a_n$ . A similar argument verifies the reverse inequality.

10. Choose  $r$  so that  $\liminf_n b_n > r > 0$ . Then, given  $\varepsilon > 0$ , there exists  $N$  such that  $a_n > a/2$  and  $b_n > r$ , and

$$c_n := (b_n - 3a_n)(b_n + 2a_n) = b_n^2 - a_n b_n - 6a_n^2 < \varepsilon$$

for every  $n > N$ . Then  $b_n - 3a_n = c_n/(b_n + 2a_n) < \varepsilon/(r + a)$ , so  $\limsup_n b_n \leq 3a$ .

11. We prove only the limsup inequality. Clearly, we may assume that  $\ell := \limsup_n a_n < +\infty$ . Let  $b_n = a_n - \ell$ . Then  $\limsup_n b_n = 0$  and we must verify that

$$\limsup_n \frac{1}{n} \sum_{j=1}^n b_j \leq 0. \quad (\dagger)$$

Let  $\varepsilon > 0$  and choose  $k$  such that  $\sup_{n \geq k} b_n < \varepsilon$ . Then

$$\begin{aligned} \limsup_n \frac{1}{n} \sum_{j=1}^n b_j &\leq \limsup_n \frac{1}{n} \sum_{j=1}^k b_j + \limsup_n \frac{1}{n} \sum_{j=k+1}^n b_j \\ &\leq \limsup_n \frac{n-k}{n} \varepsilon = \varepsilon. \end{aligned}$$

Since  $\varepsilon$  was arbitrary,  $(\dagger)$  holds.

12. Suppose that  $\liminf_n a_n^{1/n} < \liminf_n \frac{a_{n+1}}{a_n}$ . Choose  $r$  strictly between these numbers and then choose  $N$  such that  $a_n/a_{n-1} > r$  for all  $n > N$ . For such  $n$ ,

$$a_n > a_{n-1}r > a_{n-2}r^2 > \cdots > a_N r^{n-N},$$

hence

$$\liminf_n a_n^{1/n} \geq \liminf_n (a_N^{1/n} r^{1-N/n}) = r,$$

a contradiction. To evaluate  $\lim_n n/(n!)^{1/n}$  take  $a_n = n^n/n!$  and calculate

$$\frac{a_{n+1}}{a_n} = \left( \frac{n+1}{n} \right)^n \rightarrow e.$$