Chapter 2

Solutions

1. (a) By contradiction. Suppose $\sqrt{3}$ is rational, so $\sqrt{3} = \frac{m}{n}$ where m, n are integers and $\frac{m}{n}$ is in lowest terms. Squaring, we get $m^2 = 3n^2$. Thus m^2 is a multiple of 3, and so by Example 1.3, m is a multiple of 3. This means $m = 3k$ for some integer k. Then $3n^2 = m^2 = 9k^2$, so $n^2 = 3k^2$. Therefore n^2 is a multiple of 3, hence so is n. We have now shown that both m and n are multiples of 3. But $\frac{m}{n}$ is in lowest terms, so this is a contradiction. Therefore $\sqrt{3}$ is irrational.

(b) By contradiction again. Suppose $\sqrt{3} = r + s\sqrt{2}$ with r, s rational. Squaring, we get $3 = r^2 + 2s^2 + 2rs\sqrt{2}$. If $rs \neq 0$ this gives $\sqrt{2} = \frac{3-r^2-2s^2}{2rs}$. Since r, s are rational this implies that $\sqrt{2}$ is rational, which is a contradiction. Hence $rs = 0$. If $s = 0$ then $r^2 = 3$, so $r = \sqrt{3}$, contradicting the fact that $\sqrt{3}$ is irrational by (a). Therefore $r = 0$ and $3 = 2s^2$. Writing $s = \frac{m}{n}$ in lowest terms, we have $3n^2 = 2m^2$. Now the proof of Proposition 2.3 shows that m and n must both be even, which is a contradiction.

2. (a) Suppose $x = \sqrt{2} + \sqrt{3/2}$ is rational. Then $x^2 = 2 + \frac{3}{2} + 2\sqrt{3}$, hence $\sqrt{3} = \frac{1}{2}(x^2 - \frac{7}{2})$. As x is rational this implies $\sqrt{3}$ is rational, a contradiction by Q1(a). Hence $\sqrt{2} + \sqrt{3/2}$ is irrational.

(b) By (a), $1 + \sqrt{2} + \sqrt{3/2}$ is the sum of a rational and an irrational, hence is irrational by Proposition 2.4(i).

(c) We have $2\sqrt{18} - 3\sqrt{8} + \sqrt{4} = 6\sqrt{2} - 6\sqrt{2} + 2 = 2$, which is rational.

(d) Let $x = \sqrt{2} + \sqrt{3} + \sqrt{5}$. Then $(x - \sqrt{2})^2 = (\sqrt{3} + \sqrt{5})^2$, which gives $x^2 - 6 - 2\sqrt{15} = 2x\sqrt{2}$. Squaring again, $(x^2 - 6)^2 + 60 - 4(x^2 - 6)\sqrt{15} = 8x^2$, hence $\sqrt{15} = ((x^2 - 6)^2 - 8x^2 + 60)/4(x^2 - 6)$. Therefore if x is rational, then so is $\sqrt{15}$. But $\sqrt{15}$ is irrational by the hint in the question. Hence so is x.

(e) This is sneaky one. Observe that $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$, so $\sqrt{5 + 2\sqrt{6}} = \sqrt{2} + \sqrt{3}$, and hence $\sqrt{2} + \sqrt{3} - \sqrt{5 + 2\sqrt{6}} = 0$, which is rational.

3. (a) True: if $x = m/n$ and $y = p/q$ are rational, so is $xy = mp/nq$.

(b) False: for a counterexample take the irrationals $\sqrt{2}$ and $-\sqrt{2}$. Their product is -2, which is rational.

(c) False: the product of the two irrationals $\sqrt{2}$ and $1+\sqrt{2}$ is $\sqrt{2}+2$, which is irrational.

(d) True: we prove it by contradiction. Suppose there is a rational $a \neq 0$ and an irrational b such that $c = ab$ is rational. Then $b = \frac{c}{a}$, and since a and c are rational, this implies that b is rational, a contradiction.

4. (a) Let $c = \frac{x+a}{x+b}$ and suppose c is rational. Then $x+a = c(x+b)$, which gives $x(c-1) = a - bc$. If $c \neq 1$ then $x = \frac{a-bc}{c-1}$, which is rational since a, b, c are rational. As x is irrational, this implies that $c = 1$, hence $a = b$.

(b) Let $c = \frac{x^2+x+\sqrt{2}}{y^2+y+\sqrt{2}}$ and suppose c is rational. Multiplying up gives $\sqrt{2}(c-1) = x^2+x-cy^2-cy$. If $c \neq 1$ this gives $\sqrt{2} = \frac{x^2+x-cy^2-cy}{c-1}$, which is rational. Hence $c = 1$, which implies $x^2+x = y^2+y$. This yields $(x-y)(x+y+1) = 0$, hence either $x = y$ or $x+y = -1$.

5. By contradiction. Let $\alpha = \sqrt{2}+\sqrt{n}$, and suppose α is rational. Then $\alpha - \sqrt{2} = \sqrt{n}$. Squaring both sides, $\alpha^2+2-2\alpha\sqrt{2} = n$, so $2\alpha\sqrt{2} = \alpha^2+2-n$. Since clearly $\alpha \neq 0$, we can divide through by 2α to get $\sqrt{2} = (\alpha^2+2-n)/2\alpha$. As α is rational, this implies that $\sqrt{2}$ is rational, which is a contradiction. Hence α is irrational.

6. Let a and b be two different real numbers with $b > a$. Choose a positive integer n such that $b - a > \frac{1}{n}$. Then there is a rational of the form $\frac{m}{n}$ lying between a and b.

Also, choose a positive integer m such that $b - a > \frac{\sqrt{2}}{m}$. Then there is a number of the form $\frac{k\sqrt{2}}{m}$ lying between a and b, where k is an integer; by Proposition 2.4(ii), $\frac{k\sqrt{2}}{m}$ is irrational unless $k = 0$, in which case $a < 0$ and $b > 0$ and we apply the above argument replacing a by 0.

7. Let $a_n = \sqrt{n-2}+\sqrt{n+2}$. Then $a_n^2 = (n-2)+(n+2)+2\sqrt{(n-2)(n+2)} = 2n+2\sqrt{n^2-4}$. We are given that a_n is an integer. This implies $\sqrt{n^2-4}$ is rational. By the hint given, this means that n^2-4 must be a perfect square, i.e. $n^2-4 = m^2$ for some integer m. Then $n^2-m^2 = 4$. Staring at the list of squares $0, 1, 4, 9, 16, \ldots$, we see that the only way the two squares n^2, m^2 can differ by 4 is to have $n^2 = 4, m^2 = 0$. Hence $n = 2$, so $a_n = 2$.

8. Let r, b, g be the numbers of red, blue and green salamanders at some point in time. If a red and a green meet, these numbers change to $r-1, b+2, g-1$, so the difference between b and r is increased by 3; if two reds meet, the numbers change to $r-2, b+1, g+1$, and again the difference between b and r increases by 3; and so on — you can easily check that whenever two salamanders meet, the difference between b and r either stays the same, or increases or decreases by 3. Initially, $r-b$ is $15-7=8$. This cannot be changed by adding and subtracting multiples of 3 to $30-0=30$. Hence it is not possible for all the salamanders to be red at some point.